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AC Optimal Power Flow: a Conic Programming relaxation and an iterative MILP scheme for Global Optimization

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Abstract

We address the issue of computing a global minimizer of the AC Optimal Power Flow problem. We introduce valid inequalities to strengthen the Semidefinite Programming relaxation, yielding a novel Conic Programming relaxation. Leveraging these Conic Programming constraints, we dynamically generate Mixed-Integer Linear Programming (MILP) relaxations, whose solutions asymptotically converge to global minimizers of the AC Optimal Power Flow problem. We apply this iterative MILP scheme on the IEEE PES PGLib [2] benchmark and compare the results with two recent Global Optimization approaches.

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Keywords ACOPF, Global Optimization, Semidefinite Programming, Mixed-Integer Linear Programming.

1 Introduction

1.1 Motivation and related works

The Alternating-Current Optimal Power Flow (ACOPF) is a seminal optimization problem related to the dispatching of electricity in a power network. The authorship of this problem is attributed to Carpentier [6], who introduced it in 1962 as "Economic Dispatch". Since then, this problem has not only interested the Power Systems research community, but also the community of Operations Research and Mathematical Programming [5, 28]. Indeed, ACOPF was identified as a challenging and fruitful application of Nonlinear Programming (NLP) and Global Optimization methods. Thanks to Interior-Point algorithms, developed since the 1990s, the computation of ACOPF feasible solutions and local minimizers is accessible, even for instances of several thousand nodes [33]. To bound the optimality gap of feasible solutions found by such NLP algorithms, several convex relaxations have been introduced during the past decade. A review of the numerous relaxation techniques for the ACOPF problem is available in [28]. Leveraging NLP algorithms and convex relaxations techniques, several approaches emerged to solve the ACOPF problem to global optimality. We gather these works in four different categories.

- **Relaxation Strengthening and Bound Tightening.** Strengthening the classical convex relaxations [28] such as the rank relaxation helps improving the corresponding lower bounds. This strengthening is possible through additional valid inequalities coming from the polar formulation of the ACOPF problem [10, 19, 20] or derived from the Reformulation-Linearization Technique (RLT) [31]. Feasibility-Based Bound Tightening (FBBT) and Optimization-Based Bound Tightening (OBBT) techniques [3], the latter being based on the value of a known feasible solution, are also known to be particularly efficient for the ACOPF problem [9, 32]. Even if these methods do not have a guarantee of convergence towards a global solution, the aforementioned articles report that they significantly reduce the optimality gap and even close the gap for some instances.
- **Moment-Sum-of-Squares hierarchy.** The celebrated Moment-Sum-of-Squares hierarchy of relaxations for polynomial optimization problems [23] has been applied to the ACOPF problem in several works [14, 13, 27, 29]. The convergence of the relaxations' values towards the optimal value of the ACOPF problem is proven, at the price of the rapidly increasing size and computational cost of the resulting convex relaxations. In practice, only the first and second order relaxations are solvable for medium-scale ACOPF instances, using the sparse variant of the Moment-Sum-of-Squares hierarchy [24].

- **Spatial Branch-and-Bound.** Other Global Optimization approaches for the ACOPF problem follow Spatial Branch-and-Bound schemes [4]. To obtain a lower bound at each node of the exploration tree, these algorithms may use a Second-Order Cone Programming (SOCP) relaxation [21], a Quadratically Constrained Programming (QCP) relaxation [12] or a Semidefinite Programming (SDP) relaxation [7].
- **Piecewise convex relaxations.** Rather than implementing a Spatial Branch-and-Bound algorithm from scratch, an alternative is to encode branching decisions via binary variables and use an off-the-shelf Mixed-Integer Programming solver. This leads to piecewise convex relaxations, which can be iteratively refined. This is the approach followed in [25], leading to convex Mixed-Integer Quadratically Constrained Programming (MIQCP) problems.

1.2 Contributions and organization of the paper

In this quest towards Global Optimization, our contributions are manifold:

- We add valid inequalities to strengthen the SDP relaxation, which yields a Conic Programming relaxation. These valid inequalities dominate the *lifted nonlinear cuts* introduced in [10] for the same purpose.
- Leveraging the Conic Programming constraints, we propose a Global Optimization algorithm that proceeds by solving a sequence of dynamically generated piecewise linear relaxations, i.e., Mixed-Integer Linear Programming (MILP) problems. Contrary to [25], a previous paper using piecewise relaxations for the ACOPF problem, we do not use MIQCP but MILP models, which integrate cuts from the conic relaxation.
- We apply this algorithm on the IEEE PES PGLib [2] benchmark and compare the optimality gaps with two recent Global Optimization approaches [13, 32] that use this reference benchmark.

In Section 2, we present the ACOPF problem and an equivalent reformulation of it. Section 3 introduces our valid inequalities, the resulting Conic Programming relaxation, and the Bound Tightening procedure that we apply. The iterative MILP scheme is presented in Section 4 and the numerical experiments in Section 5.

1.3 Mathematical notation

For any complex number $x \in \mathbb{C}$, $x^* = \mathsf{Re}(x) - \mathbf{i}\,\mathsf{Im}(x)$ is its complex conjugate, $|x|$ is its magnitude and $\angle x$ its argument. We denote by $\mathbb{C}^{n\times n}$ the $\mathbb{C}$-vector space of $n \times n$ matrices with complex entries. We denote by $(E_{ab})_{ab}$ the canonical basis of this $\mathbb{C}$-vector space. For any matrix $M \in \mathbb{C}^{n\times n}$, its Hermitian transpose is M^{H}, defined such as $M^{\mathsf{H}}_{ab} = M^*_{ba}$ for all $b, a \in \{1, \dots, n\}$. The $\mathbb{R}$-vector space of Hermitian matrices, $\mathbb{H}_n \subset \mathbb{C}^{n\times n}$, is the set of matrices $M \in \mathbb{C}^{n\times n}$ such that (s.t.) $M = M^{\mathsf{H}}$.

2 Mathematical Programming formulations for the ACOPF

2.1 Original formulation

A power grid is a network of buses interconnected by lines. We give an arbitrary orientation to each line, so as to distinguish its two extremities. Hence, the grid is modelled as a directed graph $\mathcal{N} = (\mathcal{B}, \mathcal{L})$ with size $n = |\mathcal{B}|$. The set $\mathcal{L}$ is s.t. $\mathcal{L} \cap \mathcal{L}^R = \emptyset$, where $\mathcal{L}^R$ is the set of couples (b, a) s.t. $(a, b) \in \mathcal{L}$. A line $\ell \in \mathcal{L}$ is described by a couple (b, a) s.t. $b \in \mathcal{B}$ is the "from" bus (denoted by f), $a \in \mathcal{B}$ is the "to" bus (denoted by t). Electricity generating units are located at several buses in the network. We denote by $\mathcal{G}_b$ the set of generators located at bus $b \in \mathcal{B}$. The set of all generators is $\mathcal{G} = \bigcup_{b\in\mathcal{B}} \mathcal{G}_b$, whose cardinality is $m = |\mathcal{G}|$. The parameters of the ACOPF problem are described in Table 1.

Table 1 Parameters of the ACOPF problem

Parameters	Index set	Description
$c_{1g} \in \mathbb{R}, c_{2g} \in \mathbb{R}_+$	$g \in \mathcal{G}$	Cost parameters
$\underline{s}_g, \bar{s}_g \in \mathbb{C}$	$g \in \mathcal{G}$	Power injection bounds
$\underline{v}_b, \bar{v}_b \in [0, 2]$	$b \in \mathcal{B}$	Normalized voltage bounds
$S^{\mathsf{d}}_b \in \mathbb{C}$	$b \in \mathcal{B}$	Power demand
$Y^s_b \in \mathbb{C}$	$b \in \mathcal{B}$	Shunt admittance
$Y^{\mathsf{ff}}_{ba}, Y^{\mathsf{ft}}_{ba}, Y^{\mathsf{tf}}_{ba}, Y^{\mathsf{tt}}_{ba} \in \mathbb{C}$	$(b, a) \in \mathcal{L}$	Line impedance coefficients
$\underline{\theta}_{ba}, \bar{\theta}_{ba} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$(b, a) \in \mathcal{L} \cup \mathcal{L}^R$	Angle difference limits

In the (S,V) formulation [5], the ACOPF problem has two types of decision variables:

- for any $g \in \mathcal{G}$, $S_g \in \mathbb{C}$ is the complex power injection of generator g,
- for any $b \in \mathcal{B}$, $V_b \in \mathbb{C}$ is the complex voltage at bus b.

It is traditionally assumed that the generation cost of each generator $g \in \mathcal{G}$ is a convex quadratic form in $\mathsf{Re}(S_g)$. The objective function is the sum of the generation costs

$$\sum_{g\in\mathcal{G}} \left(c_{1g}\,\mathsf{Re}(S_g) + c_{2g}\,\mathsf{Re}(S_g)^2\right), \tag{1}$$

to be minimized. The decision variables are subject to different types of constraints:

- **Injection Limits for Generators.** For each $g \in \mathcal{G}$, we have

$$\underline{s}_g \leq S_g \leq \bar{s}_g. \tag{2}$$

  These inequalities between complex numbers designate the respective real inequalities for the real and for the imaginary parts.
- **Voltage Magnitude Limits.** For each $b \in \mathcal{B}$, the voltage at b satisfies

$$\underline{v}_b \leq |V_b| \leq \bar{v}_b. \tag{3}$$

  The ACOPF instances in the literature are scaled so that each bus has a nominal voltage normalized to 1. The lower and upper bounds respectively correspond to deviations of at most 50% around this nominal value. Introducing $\Delta_b = \bar{v}_b - \underline{v}_b$, we assume, thus, that $\underline{v}_b, \bar{v}_b \in [0,2]$ and $\Delta_b \leq 1$.
- **Power Flow Equations.** For each bus $b \in \mathcal{B}$, we define the complex matrix

$$M_b = Y^s_b E_{bb} + \sum_{a:(b,a)\in\mathcal{L}} (Y^{\mathsf{ff}}_{ba} E_{bb} + Y^{\mathsf{ft}}_{ba} E_{ba}) + \sum_{a:(b,a)\in\mathcal{L}^R} (Y^{\mathsf{tt}}_{ab} E_{bb} + Y^{\mathsf{tf}}_{ab} E_{ba}).$$

  With this notation, we write the Power Flow conservation at bus $b \in \mathcal{B}$ as

$$\sum_{g\in\mathcal{G}_b} S_g - S^{\mathsf{d}}_b = \left\langle M_b, VV^{\mathsf{H}}\right\rangle. \tag{4}$$

  Constraint (4) describes the equality between the net injection of power at b and the power transfer towards the adjacent buses.
- **Thermal Limits for Lines.** For each line $(b,a) \in \mathcal{L}$, the operational limit in terms of apparent power is

$$|(Y^{\mathsf{ff}}_{ba})^* |V_b|^2 + (Y^{\mathsf{ft}}_{ba})^* V_b V_a^*| \leq \overline{S}_{ba}. \tag{5}$$

  For $(b,a) \in \mathcal{L}^R$, this reads

$$|(Y^{\mathsf{tt}}_{ab})^* |V_b|^2 + (Y^{\mathsf{tf}}_{ab})^* V_b V_a^*| \leq \overline{S}_{ba}. \tag{6}$$

- **Line Phase Angle Difference Limits.** For any $(b,a) \in \mathcal{L} \cup \mathcal{L}^R$,

$$\underline{\theta}_{ba} \leq \angle V_b - \angle V_a \leq \bar{\theta}_{ba}. \tag{7}$$

In summary, the ACOPF problem is the following Nonconvex Optimization problem

$$\textbf{ACOPF}\left\{\begin{array}{ll} \min_{S\in\mathbb{C}^m,\,V\in\mathbb{C}^n} & \sum_{g\in\mathcal{G}} \left(c_{1g}\,\mathsf{Re}(S_g) + c_{2g}\,\mathsf{Re}(S_g)^2\right) \\ \forall\, g\in\mathcal{G} & \underline{s}_g \leq S_g \leq \bar{s}_g \\ \forall\, b\in\mathcal{B} & \underline{v}_b \leq |V_b| \leq \bar{v}_b \\ \forall\, b\in\mathcal{B} & \sum_{g\in\mathcal{G}_b} S_g - S^{\mathsf{d}}_b = \left\langle M_b, VV^{\mathsf{H}}\right\rangle \\ \forall\, (b,a)\in\mathcal{L} & |(Y^{\mathsf{ff}}_{ba})^* |V_b|^2 + (Y^{\mathsf{ft}}_{ba})^* V_b V_a^*| \leq \overline{S}_{ba} \\ \forall\, (b,a)\in\mathcal{L}^R & |(Y^{\mathsf{tt}}_{ab})^* |V_b|^2 + (Y^{\mathsf{tf}}_{ab})^* V_b V_a^*| \leq \overline{S}_{ba} \\ \forall\, (b,a)\in\mathcal{L}\cup\mathcal{L}^R & \underline{\theta}_{ba} \leq \angle V_b - \angle V_a \leq \bar{\theta}_{ba}. \end{array}\right.$$

2.2 ACOPF reformulation

▶ **Definition 1.** *A tree decomposition $\mathcal{T}$ of the graph $\mathcal{N} = (\mathcal{B}, \mathcal{L})$ is a tree where each node $k \in \mathcal{T}$ is associated with a set $\mathcal{B}_k \subset \mathcal{B}$, and satisfying the following properties*

- *The union of the subsets $\mathcal{B}_k$ equals the set $\mathcal{B}$: $\bigcup_{k\in\mathcal{T}} \mathcal{B}_k = \mathcal{B}$,*
- *For every $(b, a) \in \mathcal{L}$, there exists $k \in \mathcal{T}$ s.t $\{b, a\} \subset \mathcal{B}_k$,*
- *If $b \in \mathcal{B}_k \cap \mathcal{B}_\ell$ for any $k, \ell \in \mathcal{T}$, then $b \in \mathcal{B}_j$ for all nodes j of the tree $\mathcal{T}$ in the unique path between k and ℓ.*

We consider a given tree decomposition $\mathcal{T}$ of the graph $\mathcal{N}$, and we introduce the symmetric set $\mathcal{E} \subset \mathcal{B} \times \mathcal{B}$ of arcs defined as $\mathcal{E} = \bigcup_{k\in\mathcal{T}} \mathcal{B}_k \times \mathcal{B}_k$. As a matter of fact, the sets $\mathcal{B}_k$ are cliques of the undirected graph induced by $(\mathcal{B}, \mathcal{E})$. In this respect, the sets $\mathcal{B}_k$ are called *cliques*. We denote by $\mathbb{H}_n(\mathcal{E})$ the set of partially defined matrices W, seen as vectors indexed by $\mathcal{E}$, and s.t. $W_{ba} = W^*_{ab}$ for all $(b, a) \in \mathcal{E}$. For any $k \in \mathcal{T}$, we denote by $W_{\mathcal{B}_k,\mathcal{B}_k}$ the matrix $(W_{ba})_{(b,a)\in\mathcal{B}_k^2}$. With this notation, we reformulate (**ACOPF**) as

$$\mathbf{ACOPF_W}\begin{cases} \min_{S\in\mathbb{C}^m,\, W\in\mathbb{H}_n(\mathcal{E})} & \sum_{g\in\mathcal{G}} \left(c_{1g}\,\mathsf{Re}(S_g) + c_{2g}\,\mathsf{Re}(S_g)^2\right) \\ \forall\, g \in \mathcal{G} & \underline{s}_g \le S_g \le \bar{s}_g \\ \forall\, b \in \mathcal{B} & \underline{v}_b^2 \le W_{bb} \le \bar{v}_b^2 \\ \forall\, b \in \mathcal{B} & \sum_{g\in\mathcal{G}_b} S_g - S_b^{\mathsf{d}} = \langle M_b, W\rangle \\ \forall\, (b,a) \in \mathcal{L} & |(Y_{ba}^{\mathsf{ff}})^* W_{bb} + (Y_{ba}^{\mathsf{ft}})^* W_{ba}| \le \bar{S}_{ba} \\ \forall\, (b,a) \in \mathcal{L}^R & |(Y_{ab}^{\mathsf{tt}})^* W_{bb} + (Y_{ab}^{\mathsf{tf}})^* W_{ba}| \le \bar{S}_{ba} \\ \forall\, (b,a) \in \mathcal{L}\cup\mathcal{L}^R & \tan(\underline{\theta}_{ba})\mathsf{Re}(W_{ba}) \le \mathsf{Im}(W_{ba}) \le \tan(\bar{\theta}_{ba})\mathsf{Re}(W_{ba}) \\ \forall\, (b,a) \in \mathcal{E} & |W_{ba}|^2 = W_{bb}W_{aa} \qquad\qquad (\star) \\ \forall\, k \in \mathcal{T} & W_{\mathcal{B}_k,\mathcal{B}_k} \succeq 0. \end{cases}$$

While the clique-based SDP relaxation is well known, this clique-based reformulation of the ACOPF problem itself is not properly stated in the literature, as far as we know. Yet, we acknowledge that the proof of Theorem 2 is closely related to the developments presented in [7].

▶ **Theorem 2.** *A pair (S, W) is feasible (resp. optimal) in ($\mathbf{ACOPF_W}$) if and only if there exists $V \in \mathbb{C}^n$ s.t. (S, V) is feasible (resp. optimal) in (**ACOPF**) and $W_{ba} = V_b V_a^*$ for all $(b, a) \in \mathcal{E}$.*

Proof. We prove the equivalence for the feasibility, which also proves the equivalence for the optimality since both problems share the same objective value. We take (S, V) a feasible solution in (**ACOPF**) and we define $W \in \mathbb{H}_n(\mathcal{E})$ as $W_{ba} = V_b V_a^*$ for any $(b, a) \in \mathcal{E}$. For any $b \in \mathcal{B}$, we make the following observations:

- Since $\underline{v}_b \le |V_b| \le \bar{v}_b$, the inequalities $\underline{v}_b^2 \le |V_b|^2 \le \bar{v}_b^2$ and $\underline{v}_b^2 \le W_{bb} \le \bar{v}_b^2$ hold.
- Since $\sum_{g\in\mathcal{G}_b} S_g - S_b^{\mathsf{d}} = \langle M_b, VV^{\mathsf{H}}\rangle$ and since $M_b \in \mathbb{H}_n(\mathcal{E})$, we deduce by substitution that $\sum_{g\in\mathcal{G}_b} S_g - S_b^{\mathsf{d}} = \langle M_b, W\rangle$.

Similarly by direct substitution, we deduce that $|(Y_{ba}^{\mathsf{ff}})^* W_{bb} + (Y_{ba}^{\mathsf{ft}})^* W_{ba}| \le \bar{S}_{ba}$ for all $(b, a) \in \mathcal{L}$ and $|(Y_{ab}^{\mathsf{tt}})^* W_{bb} + (Y_{ab}^{\mathsf{tf}})^* W_{ba}| \le \bar{S}_{ba}$ for all $(b, a) \in \mathcal{L}^R$. For any $(b, a) \in \mathcal{L}\cup\mathcal{L}^R$, since $\angle W_{ba} = \angle V_b V_a^* = \angle V_b - \angle V_a$, we have $\underline{\theta}_{ba} \le \angle W_{ba} \le \bar{\theta}_{ba}$. Using that $\underline{\theta}_{ba}, \bar{\theta}_{ba} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we deduce that $\tan(\underline{\theta}_{ba})\mathsf{Re}(W_{ba}) \le \mathsf{Im}(W_{ba}) \le \tan(\bar{\theta}_{ba})\mathsf{Re}(W_{ba})$. To conclude about the feasibility of (S, W) in ($\mathbf{ACOPF_W}$), we state that $W_{\mathcal{B}_k,\mathcal{B}_k} = (V_b V_a^*)_{(b,a)\in\mathcal{B}_k^2} \succeq 0$ for all $k \in \mathcal{T}$, and that $|W_{ba}|^2 = |V_b|^2|V_a|^2 = W_{bb}W_{aa}$ for all $(b, a) \in \mathcal{E}$.

Conversely, we consider any (S, W) feasible in ($\mathbf{ACOPF_W}$). Since $W_{\mathcal{B}_k,\mathcal{B}_k} \succeq 0$ and $|W_{ba}|^2 = W_{bb}W_{aa}$ for all $(b, a) \in \mathcal{B}_k^2$, we can apply [7, Prop. 6] to state that $\mathsf{rank}\, W_{\mathcal{B}_k,\mathcal{B}_k} = 1$ for all $k \in \mathcal{T}$. By induction on the tree decomposition $\mathcal{T}$, we prove that there exists $V \in \mathbb{C}^n$ s.t. $W_{ba} = V_b(V_a)^*$ for all $(b, a) \in \mathcal{E}$. The case $|\mathcal{T}| = 1$ is trivial, since any rank-one positive semidefinite (PSD) matrix W can be written as $W = VV^{\mathsf{H}}$. We assume now that the induction hypothesis is true for any graph with a tree decomposition with size less or equal than $p \in \mathbb{N}^*$, and we consider a graph $\mathcal{N}$ with a tree decomposition $\mathcal{T}$ with size $p + 1$. We consider a leaf k of $\mathcal{T}$, $\mathcal{B}_k$ the corresponding clique, $\widetilde{\mathcal{B}} = \bigcup_{\ell\neq k} \mathcal{B}_\ell$ the union of the other cliques, and $\mathcal{C}_k = \mathcal{B}_k \setminus \widetilde{\mathcal{B}}$. By property of a tree decomposition, since k is a leaf of $\mathcal{T}$, $\mathcal{T} \setminus \{k\}$ is a tree decomposition of the graph $(\widetilde{\mathcal{B}}, \widetilde{\mathcal{E}})$, where $\widetilde{\mathcal{E}}$ denotes the edges in $\mathcal{E}$ that are not adjacent to $\mathcal{C}_k$. Applying the induction hypothesis, since $\mathcal{T} \setminus \{k\}$ has size p, there exists a complex vector $V \in \mathbb{C}^{|\widetilde{\mathcal{B}}|}$ s.t. $W_{ba} = V_b V_a^*$ for all $(b, a) \in \widetilde{\mathcal{E}}$. Additionally, since $W_{\mathcal{B}_k,\mathcal{B}_k}$ is a rank-one PSD matrix, there exists $U \in \mathbb{C}^{|\mathcal{B}_k|}$ s.t. $W_{ba} = U_b U_a^*$ for all $(b, a) \in \mathcal{B}_k \times \mathcal{B}_k$. For all $b \in \mathcal{B}_k \setminus \mathcal{C}_k$, $|V_b|^2 = W_{bb} = |U_b|^2$, since $b \in \widetilde{\mathcal{B}} \cap \mathcal{B}_k$. Hence, $|V_b| = |U_b|$ by nonnegativity of the module. Moreover, for all $(b, a) \in (\mathcal{B}_k \setminus \mathcal{C}_k)^2$, $\angle V_b - \angle V_a = \angle W_{ba} = \angle U_b - \angle U_a$, and hence, $\angle V_b - \angle U_b = \angle V_a - \angle U_a$. Defining $\mu = \angle V_b - \angle U_b$ for any $b \in \mathcal{B}_k \setminus \mathcal{C}_k$, we define $U' = e^{\mathbf{i}\mu}U$, which satisfies $U'_b = V_b$ for all $b \in \mathcal{B}_k \setminus \mathcal{C}_k$. Hence, the vector $V' \in \mathbb{C}^n$ defined as $V'_b = U'_b$ if $b \in \mathcal{B}_k$ and $V'_b = V_b$ if $b \in \widetilde{\mathcal{B}}$, is well-defined and satisfies $W_{ba} = V'_b(V'_a)^*$ for all $(b, a) \in \mathcal{E}$.

By induction, this proves that there exists a vector $V \in \mathbb{C}^n$ s.t. $W_{ba} = V_b V_a^*$ for all $(b,a) \in \mathcal{E}$. The feasibility of (S, V) in (**ACOPF**) follows by substituting W_{ba} by $V_b V_a^*$ in the constraints of ($\mathbf{ACOPF_W}$). ◀

3 Strengthening the SDP relaxation

In formulation ($\mathbf{ACOPF_W}$), Constraints ($\star$) are the only nonconvex constraints. Removing them leads to the clique-based SDP relaxation [28, 30]. Instead of merely deleting Constraints ($\star$), we add valid inequalities based on Voltage Magnitude and Phase Angle Difference bounds.

3.1 Conic Programming Outer-Approximation of Constraints ($\star$)

For all $b \in \mathcal{B}$, we introduce a variable $L_b \in [\underline{v}_b, \overline{v}_b]$ that represents the Voltage Magnitude $|V_b|$. For all $(b,a) \in \mathcal{E}$, we introduce a variable $R_{ba} \in [\underline{v}_b\underline{v}_a, \overline{v}_b\overline{v}_a]$ that stands for $|V_b||V_a|$ and is subject to

$$R_{ba} \geq \underline{v}_b L_a + \underline{v}_a L_b - \underline{v}_b\underline{v}_a \qquad R_{ba} \geq \overline{v}_b L_a + \overline{v}_a L_b - \overline{v}_b\overline{v}_a \tag{8}$$

$$R_{ba} \leq \overline{v}_b L_a + \underline{v}_a L_b - \underline{v}_a\overline{v}_b \qquad R_{ba} \leq \overline{v}_a L_b + \underline{v}_b L_a - \overline{v}_a\underline{v}_b. \tag{9}$$

For all $b \in \mathcal{B}$, we also define the following constraints

$$L_b^2 \leq R_{bb} \qquad R_{bb} = W_{bb} \tag{10}$$

$$R_{bb} + \underline{v}_b\overline{v}_b \leq (\underline{v}_b + \overline{v}_b)L_b. \tag{11}$$

Whereas Constraints (8)–(11) approximate the equality $R_{ba}^2 = W_{bb}W_{aa}$, we also need to approximate $|W_{ba}| = R_{ba}$. For this purpose, we impose for all $(b,a) \in \mathcal{E}$,

$$|W_{ba}| \leq R_{ba}. \tag{12}$$

For all $(b,a) \in \mathcal{E} \setminus (\mathcal{L} \cup \mathcal{L}^R)$, we define $\underline{\theta}_{ba} = -2\pi$ and $\bar{\theta}_{ba} = 2\pi$. In fact, we present in Section 3.3.3 how these Phase Angle Difference bounds may be tightened based on a Shortest Path algorithm. Then, we can define the angles $\phi_{ba} = \frac{\underline{\theta}_{ba} + \bar{\theta}_{ba}}{2}$ and $\delta_{ba} = \frac{\bar{\theta}_{ba} - \underline{\theta}_{ba}}{2}$ for any $(b,a) \in \mathcal{E}$. With this notation, the following constraints are valid for any $(b,a) \in \mathcal{E}$ s.t. $\delta_{ba} \leq \frac{\pi}{2}$:

$$\cos(\phi_{ba})\mathsf{Re}(W_{ba}) + \sin(\phi_{ba})\mathsf{Im}(W_{ba}) \geq R_{ba}\cos(\delta_{ba}). \tag{13}$$

Finally, for every $k \in \mathcal{T}$, we require that

$$R_{\mathcal{B}_k\mathcal{B}_k} = (R_{\mathcal{B}_k\mathcal{B}_k})^{\mathsf{H}} \qquad \begin{pmatrix} 1 & L_{\mathcal{B}_k}^{\mathsf{H}} \\ L_{\mathcal{B}_k} & R_{\mathcal{B}_k\mathcal{B}_k} \end{pmatrix} \succeq 0, \tag{14}$$

where $R_{\mathcal{B}_k\mathcal{B}_k}$ denotes the matrix $(R_{ba})_{(b,a)\in\mathcal{B}_k^2}$ and $L_{\mathcal{B}_k}$ denotes the vector $(L_b)_{b\in\mathcal{B}_k}$. Adding the decision vectors $L \in \mathbb{R}^n$ and $R \in \mathbb{R}^{\mathcal{E}}$ to the optimization problem ($\mathbf{ACOPF_W}$) and replacing Constraints ($\star$) by Constraints (8)–(14), we obtain a Conic Programming problem, that we denote (**R**).

▶ **Proposition 3.** *The Conic Programming problem* (**R**) *is a relaxation of* ($\mathbf{ACOPF_W}$)*.*

Proof. We prove the validity of the Constraints (8)–(14). More specifically, we prove that for any couple $(S, W) \in \mathbb{C}^m \times \mathbb{H}_n(\mathcal{E})$ feasible in ($\mathbf{ACOPF_W}$), the quadruplet (S, W, L, R) is feasible in (**R**), where L and R are defined as $L_b = \sqrt{W_{bb}}$ and $R_{ba} = |W_{ba}|$ for all $(b,a) \in \mathcal{E}$. Since the objective function is the same in (**R**) and ($\mathbf{ACOPF_W}$), this will prove that (**R**) is a relaxation of ($\mathbf{ACOPF_W}$). Since $R_{ba} = L_b L_a$ and $(L_b, L_a) \in [\underline{v}_b, \overline{v}_b] \times [\underline{v}_a, \overline{v}_a]$, the triplet (R_{ba}, L_b, L_a) satisfies the Mc Cormick inequalities [26] with respect to (w.r.t.) these bounds, i.e., Constraints (8)–(9). Constraint (10) is satisfied since $W_{bb} \in \mathbb{R}$, as (S, W) is feasible in ($\mathbf{ACOPF_W}$), yielding $R_{bb} = |W_{bb}| = W_{bb} = L_b^2$. Constraint (11) also being a Mc Cormick constraint (for $b = a$), it is satisfied by (R_{bb}, L_b), as $R_{bb} = L_b^2$. Constraint (12) just follows from the definition of $R_{ba} = |W_{ba}|$. For any $(b,a) \in \mathcal{E}$, we define $\theta_{ba} = \angle W_{ba}$; considering the definition of ϕ_{ba} and δ_{ba}, we notice that $|\theta_{ba} - \phi_{ba}| \leq \delta_{ba}$. For this reason, if $\delta_{ba} \leq \frac{\pi}{2}$, we obtain $\cos(|\theta_{ba} - \phi_{ba}|) \geq \cos(\delta_{ba})$, as cos is decreasing over $[0, \frac{\pi}{2}]$. Using the parity of cos, and multiplying by $R_{ba} \geq 0$, we obtain $R_{ba}\cos(\theta_{ba} - \phi_{ba}) \geq R_{ba}\cos(\delta_{ba})$ Moreover, $R_{ba}\cos(\theta_{ba} - \phi_{ba}) = |W_{ba}|(\cos(\phi_{ba})\cos(\theta_{ba}) + \sin(\phi_{ba})\sin(\theta_{ba})) = \cos(\phi_{ba})\mathsf{Re}(W_{ba}) + \sin(\phi_{ba})\mathsf{Im}(W_{ba})$, explaining that (R_{ba}, W_{ba}) satisfies Constraint (13), whenever δ. Finally, Constraint (14) just follows from the equalities $R_{ba} = |W_{ba}| = |W_{ab}| = R_{ab}$ and $\begin{pmatrix} 1 & L_{\mathcal{B}_k}^{\mathsf{H}} \\ L_{\mathcal{B}_k} & R_{\mathcal{B}_k\mathcal{B}_k} \end{pmatrix} = \begin{pmatrix} 1 \\ L_{\mathcal{B}_k} \end{pmatrix} \begin{pmatrix} 1 & L_{\mathcal{B}_k}^{\mathsf{H}} \end{pmatrix} \succeq 0$. ◀

By construction, the relaxation (**R**) is tighter than the clique-based SDP relaxation, the value of which equals the value of the standard SDP relaxation [15], also known as rank relaxation. The following theorem shows how Constraints (8)–(13) help having $|W_{ba}|^2 \approx W_{bb}W_{aa}$ when the Voltage Magnitude and Phase Angle Difference intervals are small. We recall the notation $\Delta_b = \bar{v}_b - \underline{v}_b$ and that we assume $\Delta_b \leq 1$ throughout the paper.

▶ **Theorem 4.** *For any $(b,a) \in \mathcal{E}$, the following statements hold*

- *Under Constraints (8)–(9), we have $|R_{ba}^2 - L_a^2 L_b^2| \leq 9\Delta_b\Delta_a$.*
- *Under Constraints (10)–(11), we have $|W_{bb}W_{aa} - L_b^2 L_a^2| \leq (\Delta_b + \Delta_a)^2$.*
- *Under Constraints (12)–(13), if $\delta_{ba} \leq \frac{\pi}{2}$, we have $|\,|W_{ba}|^2 - R_{ba}^2| \leq 16\delta_{ba}^2$.*

Therefore, if Constraints (8)–(13) are satisfied and $\delta_{ba} \leq \frac{\pi}{2}$, then $|\,|W_{ba}|^2 - W_{bb}W_{aa}| \leq 9\Delta_b\Delta_a + (\Delta_b+\Delta_a)^2 + 16\delta_{ba}^2$.

Proof. First, we take any $(b,a) \in \mathcal{E}$ and we define a tuplet (W,L,R) satisfying Constraints (8)–(9). We define $a_1 = \underline{v}_b L_a + \underline{v}_a L_b - \underline{v}_b\underline{v}_a$ and we notice that $L_bL_a - a_1 = (L_b - \underline{v}_b)(L_a - \underline{v}_a) \in [0, \Delta_b\Delta_a]$, since $L_b - \underline{v}_b \in [0, \Delta_b]$ and $L_a - \underline{v}_a \in [0, \Delta_a]$. Hence, $a_1 \in [L_bL_a - \Delta_b\Delta_a, L_bL_a]$. Similarly, defining $a_2 = \bar{v}_bL_a + \bar{v}_aL_b - \bar{v}_b\bar{v}_a, a_3 = \bar{v}_bL_a + \underline{v}_aL_b - \underline{v}_a\bar{v}_b$ and $a_4 = \bar{v}_aL_b + \underline{v}_bL_a - \bar{v}_a\underline{v}_b$, we can prove that $a_2 \in [L_bL_a - \Delta_b\Delta_a, L_bL_a]$, $a_3 \in [L_bL_a, L_bL_a + \Delta_b\Delta_a]$ and $a_4 \in [L_bL_a, L_bL_a + \Delta_b\Delta_a]$. According to Constraints (8)–(9), $R_{ba} \in [\max(a_1,a_2), \min(a_3,a_4)]$, which proves that $R_{ba} \in [L_bL_a - \Delta_b\Delta_a, L_bL_a + \Delta_b\Delta_a]$. We square the inequalities $0 \leq R_{ba} \leq L_bL_a + \Delta_b\Delta_a$ to obtain

$$R_{ba}^2 \leq L_b^2L_a^2 + \Delta_b\Delta_a(2L_bL_a + \Delta_b\Delta_a) \leq L_b^2L_a^2 + 9\Delta_b\Delta_a,$$

the last inequality following from $L_b \leq \bar{v}_b \leq 2$, $L_a \leq \bar{v}_a \leq 2$ and $0 \leq \Delta_b\Delta_a \leq 1$. Squaring the inequalities $0 \leq L_bL_a \leq R_{ba} + \Delta_b\Delta_a$, we deduce that $L_b^2L_a^2 \leq R_{ba}^2 + \Delta_b\Delta_a(2R_{ba} + \Delta_b\Delta_a) \leq R_{ba}^2 + 9\Delta_b\Delta_a$ since $R_{ba} \leq \bar{v}_b\bar{v}_a \leq 4$. Consequently,

$$|R_{ba}^2 - L_a^2L_b^2| \leq 9\Delta_b\Delta_a. \tag{15}$$

Second, we take any tuplet (W,L,R) satisfying Constraints (10)–(11) for b and a. We notice that the maximum of the quadratic form $(\underline{v}_b + \bar{v}_b)X - X^2 - \underline{v}_b\bar{v}_b$ is attained for $X = \frac{\underline{v}_b + \bar{v}_b}{2}$ with value $\frac{(\underline{v}_b+\bar{v}_b)^2}{4} - \underline{v}_b\bar{v}_b = \frac{\Delta_b^2}{4}$. Hence, $(\underline{v}_b + \bar{v}_b)L_b - L_b^2 - \underline{v}_b\bar{v}_b \leq \frac{\Delta_b^2}{4}$. Constraint (11) yielding $R_{bb} + \underline{v}_b\bar{v}_b \leq (\underline{v}_b + \bar{v}_b)L_b$, we deduce that $R_{bb} - L_b^2 \leq \Delta_b^2/4$. As $R_{bb} \geq 0$, we have $0 \leq R_{bb} \leq L_b^2 + \Delta_b^2/4$. Applying the same reasoning for a, we have $0 \leq R_{aa} \leq L_a^2 + \Delta_a^2/4$. Multiplying both sets of inequalities together, we obtain

$$0 \leq R_{bb}R_{aa} \leq L_b^2L_a^2 + L_b^2\frac{\Delta_a^2}{4} + L_a^2\frac{\Delta_b^2}{4} + \frac{\Delta_b^2}{4}\frac{\Delta_a^2}{4} \leq L_b^2L_a^2 + \Delta_a^2 + \Delta_b^2 + 2\Delta_b\Delta_a \leq L_b^2L_a^2 + (\Delta_b + \Delta_a)^2, \tag{16}$$

using that $L_b, L_a \in [0,2]$ and $\Delta_b, \Delta_a \in [0,1]$. As Constraint (10) yields $L_b^2 \leq R_{bb}$ and $L_a^2 \leq R_{aa}$, we deduce that $L_b^2L_a^2 \leq R_{bb}R_{aa}$ and finally, since $R_{bb} = W_{bb}$ and $R_{aa} = W_{aa}$,

$$|W_{bb}W_{aa} - L_b^2L_a^2| \leq (\Delta_b + \Delta_a)^2. \tag{17}$$

Third, we take any tuplet (W,L,R) satisfying Constraints (12)–(13). We consider $(b,a) \in \mathcal{E}$ s.t. $\delta_{ba} \leq \frac{\pi}{2}$. We write W_{ba} as $|W_{ba}|e^{i\theta}$ and Constraint (13), which is applicable since $\delta_{ba} \leq \frac{\pi}{2}$, yields $|W_{ba}|(\cos(\phi_{ba})\cos(\theta) + \sin(\phi_{ba})\sin(\theta)) \geq R_{ba}\cos(\delta_{ba})$. This may be written as $|W_{ba}|\cos(\phi_{ba} - \theta) \geq R_{ba}\cos(\delta_{ba})$. This implies that $|W_{ba}| \geq R_{ba}\cos(\delta_{ba})$, and thus $|W_{ba}|^2 \geq R_{ba}^2\cos(\delta_{ba})^2$. As $|W_{ba}|^2 \leq R_{ba}^2$, according to Constraint (12), we have

$$0 \leq R_{ba}^2 - |W_{ba}|^2 \leq R_{ba}^2(1 - \cos(\delta_{ba})^2) = R_{ba}^2\sin(\delta_{ba})^2.$$

Using that $R_{ba} \leq 4$ and that $\sin(\delta_{ba})^2 \leq \delta_{ba}^2$, we obtain

$$\big||W_{ba}|^2 - R_{ba}^2\big| \leq 16\delta_{ba}^2. \tag{18}$$

As a conclusion, for any tuplet (W,L,R) satisfying Constraints (8)–(13), we deduce from Equations (15), (17) and (18) that

$$\big||W_{ba}|^2 - W_{bb}W_{aa}\big| \leq \big||W_{ba}|^2 - R_{ba}^2\big| + |R_{ba}^2 - L_b^2L_a^2| + |L_b^2L_a^2 - W_{bb}W_{aa}| \leq 9\Delta_b\Delta_a + (\Delta_b + \Delta_a)^2 + 16\delta_{ba}^2.$$

◀

3.2 Connections to previous works

Previous works in the Power Systems community proposed valid inequalities to strengthen the SDP relaxation of the ACOPF problem [19, 20, 10]. In [10], the authors show that these valid inequalities are all dominated by the inequalities [10, (36a) and (36b)]. Using the parameter $v_b^\sigma = \underline{v}_b + \overline{v}_b$, the inequalities [10, (36a) and (36b)] read, with our notation,

$$v_b^\sigma v_a^\sigma \left(\cos(\phi_{ba})\mathsf{Re}(W_{ba}) + \sin(\phi_{ba})\mathsf{Im}(W_{ba})\right) - \overline{v}_a \cos(\delta_{ba}) v_a^\sigma W_{bb} - \overline{v}_b \cos(\delta_{ba}) v_b^\sigma W_{aa} \geq \overline{v}_b\overline{v}_a \cos(\delta_{ba})(\underline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a) \quad (\dagger)$$

$$v_b^\sigma v_a^\sigma \left(\cos(\phi_{ba})\mathsf{Re}(W_{ba}) + \sin(\phi_{ba})\mathsf{Im}(W_{ba})\right) - \underline{v}_a \cos(\delta_{ba}) v_a^\sigma W_{bb} - \underline{v}_b \cos(\delta_{ba}) v_b^\sigma W_{aa} \geq -\underline{v}_b\underline{v}_a \cos(\delta_{ba})(\underline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a). \quad (\ddagger)$$

The following Proposition states that Constraints (8)–(13), that we introduce here to strengthen the SDP relaxation, dominate Equations (†)–(‡).

▶ **Proposition 5.** *For any $(b,a) \in \mathcal{L} \cup \mathcal{L}^R$, for any quadruplet $(\mathsf{Re}(W_{ba}), \mathsf{Im}(W_{ba}), W_{bb}, W_{aa})$ s.t. there exists $L_b, L_a, R_{ba} \in \mathbb{R}_+$ s.t. Constraints (8)–(13) are satisfied, then the quadruplet $(\mathsf{Re}(W_{ba}), \mathsf{Im}(W_{ba}), W_{bb}, W_{aa})$ satisfies (†)–(‡).*

Proof. We take any $(b,a) \in \mathcal{L} \cup \mathcal{L}^R$ and any quadruplet $(\mathsf{Re}(W_{ba}), \mathsf{Im}(W_{ba}), W_{bb}, W_{aa})$ s.t. there exists $L_b, L_a, R_{ba} \in \mathbb{R}_+$ s.t. Constraints (8)–(13) are satisfied. Constraints (10)–(11) applied for b and a yields

$$v_b^\sigma L_b \geq W_{bb} + \overline{v}_b\underline{v}_b \quad (19)$$

$$v_a^\sigma L_a \geq W_{aa} + \overline{v}_a\underline{v}_a. \quad (20)$$

First, we combine Equations (19)–(20) with $R_{ba} \geq \overline{v}_a L_b + \overline{v}_b L_a - \overline{v}_b\overline{v}_a$ from Constraint (8), that we multiply by $v_b^\sigma v_a^\sigma \geq 0$, to deduce that $v_b^\sigma v_a^\sigma R_{ba} \geq \overline{v}_a v_a^\sigma W_{bb} + \overline{v}_b v_b^\sigma W_{aa} + \overline{v}_a v_a^\sigma \overline{v}_b \underline{v}_b + \overline{v}_b v_b^\sigma \overline{v}_a \underline{v}_a - v_b^\sigma v_a^\sigma \overline{v}_b\overline{v}_a$ and, thus,

$$v_b^\sigma v_a^\sigma R_{ba} - \overline{v}_a v_a^\sigma W_{bb} - \overline{v}_b v_b^\sigma W_{aa} \geq \overline{v}_a v_a^\sigma \overline{v}_b\underline{v}_b + \overline{v}_b v_b^\sigma \overline{v}_a \underline{v}_a - v_b^\sigma v_a^\sigma \overline{v}_b\overline{v}_a = \overline{v}_b\overline{v}_a(\underline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a), \quad (21)$$

as $\overline{v}_a v_a^\sigma \overline{v}_b\underline{v}_b + \overline{v}_b v_b^\sigma \overline{v}_a\underline{v}_a - v_b^\sigma v_a^\sigma \overline{v}_b\overline{v}_a = \overline{v}_b\overline{v}_a(\underline{v}_b\underline{v}_a + \underline{v}_b\overline{v}_a + \underline{v}_b\underline{v}_a + \overline{v}_b\underline{v}_a - \underline{v}_b\underline{v}_a - \underline{v}_b\overline{v}_a - \overline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a) = \overline{v}_b\overline{v}_a(\underline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a)$. Multiplying Equation (21) by $\cos(\delta_{ba}) \geq 0$, we have

$$v_b^\sigma v_a^\sigma \cos(\delta_{ba}) R_{ba} - \overline{v}_a \cos(\delta_{ba}) v_a^\sigma W_{bb} - \overline{v}_b \cos(\delta_{ba}) v_b^\sigma W_{aa} \geq \overline{v}_b\overline{v}_a \cos(\delta_{ba})(\underline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a). \quad (22)$$

Multiplying Constraint (13) by $v_b^\sigma v_a^\sigma \geq 0$ yields $v_b^\sigma v_a^\sigma(\cos(\phi_{ba})\mathsf{Re}(W_{ba}) + \sin(\phi_{ba})\mathsf{Im}(W_{ba})) \geq v_b^\sigma v_a^\sigma \cos(\delta_{ba}) R_{ba}$; combining this with (22), we deduce Equation (†). We underline that Constraint (13) is indeed applicable since $\delta_{ba} \leq \frac{\pi}{2}$, as $(b,a) \in \mathcal{L} \cap \mathcal{L}^R$ (see Table 1).

Second, we combine the Equations (19)–(20) with $R_{ba} \geq \underline{v}_a L_b + \underline{v}_b L_a - \underline{v}_b\underline{v}_a$ from Constraint (8) that we multiply by $v_b^\sigma v_a^\sigma \geq 0$, to obtain $v_b^\sigma v_a^\sigma R_{ba} \geq \underline{v}_a v_a^\sigma W_{bb} + \underline{v}_b v_b^\sigma W_{aa} + \underline{v}_a v_a^\sigma \overline{v}_b\underline{v}_b + \underline{v}_b v_b^\sigma \overline{v}_a\underline{v}_a - v_b^\sigma v_a^\sigma \underline{v}_b\underline{v}_a$. As $\underline{v}_a v_a^\sigma \overline{v}_b\underline{v}_b + \underline{v}_b v_b^\sigma \overline{v}_a\underline{v}_a - v_b^\sigma v_a^\sigma \underline{v}_b\underline{v}_a = \underline{v}_b\underline{v}_a(\overline{v}_b\underline{v}_a + \overline{v}_b\overline{v}_a + \underline{v}_b\overline{v}_a + \overline{v}_b\overline{v}_a - \underline{v}_b\underline{v}_a - \underline{v}_b\overline{v}_a - \overline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a) = -\underline{v}_b\underline{v}_a(\underline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a)$, we deduce that

$$v_b^\sigma v_a^\sigma R_{ba} - \underline{v}_a v_a^\sigma W_{bb} - \underline{v}_b v_b^\sigma W_{aa} \geq -\underline{v}_b\underline{v}_a(\underline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a). \quad (23)$$

Multiplying Equation (23) by $\cos(\delta_{ba}) \geq 0$, we obtain

$$v_b^\sigma v_a^\sigma \cos(\delta_{ba}) R_{ba} - \underline{v}_a \cos(\delta_{ba}) v_a^\sigma W_{bb} - \underline{v}_b \cos(\delta_{ba}) v_b^\sigma W_{aa} \geq -\underline{v}_b\underline{v}_a \cos(\delta_{ba})(\underline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a). \quad (24)$$

Multiplying Constraint (13) by $v_b^\sigma v_a^\sigma \geq 0$ yields $v_b^\sigma v_a^\sigma(\cos(\phi_{ba})\mathsf{Re}(W_{ba}) + \sin(\phi_{ba})\mathsf{Im}(W_{ba})) \geq v_b^\sigma v_a^\sigma \cos(\delta_{ba}) R_{ba}$; combining this with (24), we deduce Equation (‡). ◀

The advantage of Constraints (8)–(13) is to enforce a coupling between the convex envelopes of the quadruplets $(\mathsf{Re}(W_{ba}), \mathsf{Im}(W_{ba}), W_{bb}, W_{aa})$ involving a same index b. This coupling is realized by the additional decision vectors L and R. In Appendix A, we present an illustrative example of two quadruplets $(\mathsf{Re}(W_{ba}), \mathsf{Im}(W_{ba}), W_{bb}, W_{aa})$ and $(\mathsf{Re}(W_{bc}), \mathsf{Im}(W_{bc}), W_{bb}, W_{cc})$ satisfying Equations (†)–(‡) introduced in [10], but for which there is no vector L and R s.t. Constraints (8)–(13) are satisfied. In this respect, we can state that Constraints (8)–(13) strictly dominate Equations (†)–(‡).

3.3 Bound Tightening procedures

We use Bound Tightening procedures to reduce the interval lengths Δ_b and δ_{ba} and, thus, reduce the error bound in Theorem 4.

3.3.1 Feasibility-based Bound Tightening (FBBT)

The Power Flow limit for the line $(b,a) \in \mathcal{L}$ implicitly restricts the phase $\angle V_b V_a^*$ and, consequently, can help reduce the length of the interval $[\underline{\theta}_{ba}, \bar{\theta}_{ba}]$. Dividing the inequality $|(Y_{ba}^{\mathsf{ft}})^* V_b V_a^* + (Y_{ba}^{\mathsf{ff}})^* |V_b|^2| \leq \bar{S}_{ba}$ by $|Y_{ba}^{\mathsf{ft}} V_b V_a|$, we deduce that $|\frac{V_b V_a^*}{|V_a||V_b|} - z\frac{|V_b|}{|V_a|}| \leq R$, where $z = \frac{(Y_{ba}^{\mathsf{ff}})^*}{(Y_{ba}^{\mathsf{ft}})^*}$ and $R = \frac{\bar{S}_{ba}}{|Y_{ba}^{\mathsf{ft}} V_b V_a|}$. We notice that $u = \frac{V_b V_a^*}{|V_a||V_b|}$ is a unit complex number and has a nonnegative real part since $\angle V_b - \angle V_a \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Representing the ratio $\frac{|V_b|}{|V_a|}$ by a variable λ, we can formulate the following Convex Optimization problem

$$\begin{cases} \max_{u,\lambda} & \mathsf{Im}(u) \\ \text{s.t.} & |u - z\lambda| \leq R \\ & \mathsf{Re}(u) \geq 0 \\ & |u| \leq 1 \\ & u \in \mathbb{C}, \lambda \in [\frac{\underline{v}_b}{\bar{v}_a}, \frac{\bar{v}_b}{\underline{v}_a}]. \end{cases} \tag{25}$$

Denoting by $\bar{h}$ its value, we deduce that $\arcsin(\bar{h})$ is an upper-bound on $\angle V_b - \angle V_a$. Hence, we can set $\bar{\theta}_{ba} \leftarrow \min(\bar{\theta}_{ba}, \arcsin(\bar{h}))$ without changing the value of (**ACOPF**). If we minimize $\mathsf{Im}(u)$ under the same constraints to get a value $\underline{h}$, we can set $\underline{\theta}_{ba} \leftarrow \max(\underline{\theta}_{ba}, \arcsin(\underline{h}))$.

Similarly for any $(b,a) \in \mathcal{L}^R$, leveraging the inequality $|(Y_{ab}^{\mathsf{tf}})^* V_b V_a^* + (Y_{ab}^{\mathsf{tt}})^* |V_b|^2| \leq \bar{S}_{ba}$, we use the same procedure with $z = \frac{(Y_{ab}^{\mathsf{tt}})^*}{(Y_{ab}^{\mathsf{tf}})^*}$ and $R = \frac{S_{ba}}{|Y_{ab}^{\mathsf{tf}} V_b V_a|}$ to tighten $\underline{\theta}_{ba}$ and $\bar{\theta}_{ba}$. This type of Bound Tightening is cheap, since it requires to solve a 2-variable optimization problem for each bound.

3.3.2 Optimization-Based Bound Tightening (OBBT)

We also apply a OBBT procedure to the Conic Programming relaxation (**R**), as performed in [32] with the QCP relaxation. We use any NLP algorithm to find an ACOPF feasible solution. With the corresponding upper-bound denoted $\overline{\mathsf{obj}}$, we add the constraint $\sum_{g \in \mathcal{G}} c_{1g} \mathsf{Re}(S_g) + c_{2g} \mathsf{Re}(S_g)^2 \leq \overline{\mathsf{obj}}$ to Problem (**R**). We denote by $\mathcal{F}$ the resulting convex feasible set for the tuplet (S, W, L, R). Then, we update the following bounds:

- For the **Voltage Magnitude** at bus $b \in \mathcal{B}$, we set

$$\bar{v}_b \leftarrow \max_{(S,W,L,R) \in \mathcal{F}} L_b \tag{26}$$

$$\underline{v}_b \leftarrow \min_{(S,W,L,R) \in \mathcal{F}} L_b. \tag{27}$$

- For the **Phase Angle Difference** on line $(b,a) \in \mathcal{L}$, we compute $\bar{h}_{ba} = \max_{(S,W,L,R) \in \mathcal{F}} \mathsf{Im}(W_{ba})$ and $\underline{h}_{ba} = \min_{(S,W,L,R) \in \mathcal{F}} \mathsf{Im}(W_{ba})$ and set

$$\bar{\theta}_{ba} \leftarrow \min\left(\bar{\theta}_{ba}, \arcsin\left(\max\left(\frac{\bar{h}_{ba}}{\bar{v}_b \bar{v}_a}, \frac{\bar{h}_{ba}}{\underline{v}_b \underline{v}_a}\right)\right)\right) \tag{28}$$

$$\underline{\theta}_{ba} \leftarrow \max\left(\underline{\theta}_{ba}, \arcsin\left(\min\left(\frac{\underline{h}_{ba}}{\bar{v}_b \bar{v}_a}, \frac{\underline{h}_{ba}}{\underline{v}_b \underline{v}_a}\right)\right)\right). \tag{29}$$

3.3.3 Shortest Path algorithm to tighten Phase Angle Difference bounds

Through FBBT and OBBT, we may individually improve the bounds $\underline{\theta}_{ba}$ and $\bar{\theta}_{ba}$ for any $(b,a) \in \mathcal{E}$. To propagate the reduction of the Phase Angle Difference domains, we apply a Shortest Path algorithm. Indeed we notice that, for any $(b_0, b_t) \in \mathcal{B} \times \mathcal{B}$, for any path $b_0, b_1, \ldots, b_t$ in the graph $(\mathcal{B}, \mathcal{E})$, for any feasible solution (S, V) in (**ACOPF**), we have $\angle V_{b_t} - \angle V_{b_0} = \sum_{s=0}^{t-1} \angle V_{b_{s+1}} - \angle V_{b_s} \leq \sum_{s=0}^{t-1} \bar{\theta}_{b_{s+1} b_s}$. The Shortest Path between b_0 and b_t in the directed weighted graph $(\mathcal{B}, \mathcal{E}, \bar{\theta})$ helps finding the lowest sum $\sum_{s=0}^{t-1} \bar{\theta}_{b_{s+1} b_s}$ to update $\bar{\theta}_{b_t b_0}$. Symmetrically, we have that $\angle V_{b_t} - \angle V_{b_0} \geq \sum_{s=0}^{t-1} \underline{\theta}_{b_{s+1} b_s}$. The Shortest Path between b_0 and b_t in the directed weighted graph $(\mathcal{B}, \mathcal{E}, -\underline{\theta})$ helps improving the lower-bound on $\angle V_{b_t} - \angle V_{b_0}$ to update $\underline{\theta}_{b_t b_0}$. To compute Shortest Paths, we apply the Floyd–Warshall algorithm [11], which fits the context of a weighted directed graph, with weights of unspecified sign. May the Floyd–Warshall algorithm find a cycle with negative weight in the directed weighted graph $(\mathcal{B}, \mathcal{E}, \bar{\theta})$, it would certify the infeasibility of (**ACOPF**), since it would give a path $b_0, b_1, \ldots, b_t$ with $b_t = b_0$ and $0 = \angle V_{b_t} - \angle V_{b_0} = \sum_{s=0}^{t-1} \angle V_{b_{s+1}} - \angle V_{b_s} \leq \sum_{s=0}^{t-1} \bar{\theta}_{b_{s+1} b_s} < 0$. Similarly, finding a cycle of negative weight in $(\mathcal{B}, \mathcal{E}, -\underline{\theta})$ certifies the infeasibility of (**ACOPF**).

4 A MILP-based Global Optimization algorithm

Leveraging the Conic Programming relaxation $(\mathbf{R})$ and its solution, we generate a sequence of MILP problems whose values converge to the ACOPF value.

4.1 Linear Programming Outer-Approximations

The disadvantage of Problem $(\mathbf{R})$ is its computational cost, that is higher, due to SDP constraints, than the cost of a Linear Programming (LP) or a convex QCP relaxation. Hence, it may not be computationally efficient to solve such a relaxation at every node of an exploration tree. The idea of our approach is to solve the relaxation $(\mathbf{R})$ at the root node only, and use it to generate a LP relaxation with the same value. We denote by $x \in \mathbb{R}^N$ the decision vector $(\mathsf{Re}(S), \mathsf{Im}(S), \mathsf{Re}(W), \mathsf{Im}(W), L, R)$, and we notice that the Problem $(\mathbf{R})$ may also be seen as

$$\begin{cases} \min_{x\in\mathcal{P}} & f_0(x) \\ \forall\, j \in \{1,\dots,M\} & f_j(x) \leq 0, \end{cases} \tag{$\mathbf{R}$}$$

with $\mathcal{P} \subset \mathbb{R}^N$ being a polytope and $f_0(x), f_1(x), \dots, f_M(x)$ continuous and convex functions. In Appendix B, we detail this polytope, the functions $f_j(x)$, and show that they share a common structure: for any $j \in \{0,\dots,M\}$, there exists an affine application $\pi_j : \mathbb{R}^N \mapsto \mathbb{R}^{p_j}$ and a compact and convex set $\mathcal{U}_j \subset \mathbb{R}^{p_j}$ s.t. for all $x \in \mathcal{P}$, $f_j(x) = \max_{u\in\mathcal{U}_j} u^\top \pi_j(x)$. For any finite subset $\check{\mathcal{U}}_j \subset \mathcal{U}_j$, we define the following polyhedral function $\check{f}_j(x) = \max_{u\in\check{\mathcal{U}}_j} u^\top \pi_j(x)$. This function, called "cutting-plane model", is an underestimator of f_j. If we relax the formulation $(\mathbf{R})$ by replacing each function $f_j(x)$ by its polyhedral underestimator $\check{f}_j(x)$ related to a given finite set $\check{\mathcal{U}}_j$, we obtain the LP relaxation

$$\begin{cases} \min_{x\in\mathcal{P}} & \check{f}_0(x) \\ \forall\, j \in \{1,\dots,M\} & \check{f}_j(x) \leq 0 \end{cases} = \begin{cases} \min_{x\in\mathcal{P},\lambda\in\mathbb{R}} & \lambda \\ \forall\, u \in \check{\mathcal{U}}_0 & u^\top \pi_0(x) \leq \lambda \\ \forall\, j \in \{1,\dots,M\}, \forall\, u \in \check{\mathcal{U}}_j & u^\top \pi_j(x) \leq 0. \end{cases} \tag{$\mathbf{R}_L$}$$

We show that based on a primal-dual solution of $(\mathbf{R})$, we can compute finite sets $\check{\mathcal{U}}_0, \dots, \check{\mathcal{U}}_M$ s.t. $\mathsf{val}(\mathbf{R}) = \mathsf{val}(\mathbf{R}_L)$. For $j \in \{1,\dots,M\}$ we define $\mathcal{K}_j$ as the convex cone generated by $\mathcal{U}_j$. We define $\mathcal{K}_0$ as the convex cone generated by $\{1\} \times \mathcal{U}_0$. We also define $\underline{\lambda}$ and $\bar{\lambda}$ as *a priori* lower and upper bounds on the value of $(\mathbf{R})$, that may be very rough estimates. We introduce a Lagrangian L function for the conic program $(\mathbf{R})$, defined for any $(x,\lambda) \in \mathcal{P} \times [\underline{\lambda}, \bar{\lambda}], \kappa = (\kappa_0, \kappa_1, \dots, \kappa_M) \in \mathcal{K}_0 \times \mathcal{K}_1 \times \cdots \times \mathcal{K}_M$ as $L(x,\lambda,\kappa) = \lambda + \kappa_0^\top \begin{pmatrix} -\lambda \\ \pi_0(x) \end{pmatrix} + \sum_{j=1}^M \kappa_j^\top \pi_j(x)$. With this definition, we see that the conic program $(\mathbf{R})$ is the min-max problem

$$\inf_{x\in\mathcal{P},\lambda\in[\underline{\lambda},\bar{\lambda}]} \; \sup_{\kappa\in\mathcal{K}_0\times\mathcal{K}_1\times\cdots\times\mathcal{K}_M} L(x,\lambda,\kappa).$$

We define the concave function $D(\kappa) = \inf_{x\in\mathcal{P},\lambda\in[\underline{\lambda},\bar{\lambda}]} L(x,\lambda,\kappa) \in \mathbb{R} \cup \{-\infty\}$, and the dual optimization problem

$$\sup_{\kappa\in\mathcal{K}_0\times\mathcal{K}_1\times\cdots\times\mathcal{K}_M} D(\kappa). \tag{30}$$

▶ **Proposition 6.** *There is no duality gap between Problem* $(\mathbf{R})$ *and Problem* (30)*, i.e., they share the same value. Moreover, if Problem* (30) *has an optimal solution* $\kappa^* \in \mathcal{K}_0 \times \mathcal{K}_1 \times \cdots \times \mathcal{K}_M$*, written as* $\kappa^* = (\eta_0 v_0^*, \eta_1 u_1^*, \dots \eta_M u_M^*)$ *with*

- $\eta_j \in \mathbb{R}_+$ *for all* $j \in \{0,1,\dots,M\}$*,*
- $v_0^* = (1, u_0^*)$ *with* $u_0^* \in \mathcal{U}_0$*, and* $u_j^* \in \mathcal{U}_j$ *for all* $j \in \{0,1,\dots,M\}$*,*

then the definition of the finite sets $\check{\mathcal{U}}_j = \{u_j^*\}$ *yields a LP relaxation* $(\mathbf{R}_L)$ *that satisfies* $\mathsf{val}(\mathbf{R}_L) = \mathsf{val}(\mathbf{R})$.

Proof. The absence of duality gap follows from the Sion min-max theorem [22], since

- The primal optimization set $\mathcal{P} \times [\underline{\lambda}, \bar{\lambda}]$ is convex and compact,
- The dual optimization set $\mathcal{K}_0 \times \mathcal{K}_1 \times \cdots \times \mathcal{K}_M$ is convex,
- The Lagrangian L is continuous and convex w.r.t. (x,λ) for any $\kappa \in \mathcal{K}_0 \times \mathcal{K}_1 \times \cdots \times \mathcal{K}_M$, and,
- The Lagrangian L is continuous and concave w.r.t. κ for any $(x,\lambda) \in \mathcal{P} \times [\underline{\lambda}, \bar{\lambda}]$.

Then, we assume that Problem (30) has an optimal solution $\kappa^* \in \mathcal{K}_0 \times \mathcal{K}_1 \times \cdots \times \mathcal{K}_M$. Due to the absence of duality gap, we know that $D(\kappa^*) = \mathsf{val}(\mathbf{R})$. Writing $\kappa^* = (\eta_0(1, u_0^*), \eta_1 u_1^*, \ldots \eta_M u_M^*)$ as indicated above, we define $\check{\mathcal{U}}_j = \{u_j^*\} \subset \mathcal{U}_j$ and $\check{\mathcal{K}}_j = \mathsf{cone}(\check{\mathcal{U}}_j)$, for all $j \in \{0, \ldots, M\}$. With this definition, $(\mathbf{R}_L)$ reads

$$\inf_{x\in\mathcal{P},\lambda\in[\underline{\lambda},\bar{\lambda}]} \quad \sup_{\kappa\in\check{\mathcal{K}}_0\times\check{\mathcal{K}}_1\times\cdots\times\check{\mathcal{K}}_M} \quad L(x, \lambda, \kappa),$$

and its dual problem is $\sup_{\kappa\in\check{\mathcal{K}}_0\times\check{\mathcal{K}}_1\times\cdots\times\check{\mathcal{K}}_M} D(\mathbf{u})$. As $\kappa^* \in \check{\mathcal{K}}_0 \times \check{\mathcal{K}}_1 \times \cdots \times \check{\mathcal{K}}_M$, we can write by weak duality that $\mathsf{val}(\mathbf{R}_L) \geq D(\kappa^*) = \mathsf{val}(\mathbf{R})$. As $(\mathbf{R}_L)$ is a relaxation of $(\mathbf{R})$, we conclude that $\mathsf{val}(\mathbf{R}_L) = \mathsf{val}(\mathbf{R})$. ◀

At the price of finding an optimal primal-dual solution $(x^*, \lambda^*, \kappa^*)$ of $(\mathbf{R})$–(30), we can build an LP relaxation with the same value. In practice, we obtain such a primal-dual solution for every tested instance.

4.2 Binary variables to encode Piecewise Linear constraints

4.2.1 Partitioning Voltage Magnitude intervals

For any $b \in \mathcal{B}$, we may want to split the interval $[\underline{v}_b, \bar{v}_b]$ in subintervals. We introduce a tree $\mathcal{J}_b$ and pairs $(\underline{v}_{bj}, \bar{v}_{bj})$ s.t. $\underline{v}_{bj} \leq \bar{v}_{bj}$ for all $j \in \mathcal{J}_b$. For r being the root node of $\mathcal{J}_b$, we have $(\underline{v}_{br}, \bar{v}_{br}) = (\underline{v}_b, \bar{v}_b)$. Denoting $\mathcal{J}_b^+(i)$ the set of the child nodes of i, the partition $[\underline{v}_{bi}, \bar{v}_{bi}] = \bigcup_{j\in\mathcal{J}_b^+(i)} [\underline{v}_{bj}, \bar{v}_{bj}]$ holds for any $i \in \mathcal{J}_b$. For any $j \in \mathcal{J}_b$, we introduce a variable $\alpha_{bj} \in \{0, 1\}$. To encode the equivalence $(\alpha_{bj} = 1) \iff (L_b \in [\underline{v}_{bj}, \bar{v}_{bj}])$ for any $j \in \mathcal{J}_b$, we impose $\alpha_{br} = 1$, and for any $j \in \mathcal{J}_b$

$$\underline{v}_{bj}\alpha_{bj} + (1 - \alpha_{bj})\underline{v}_b \leq L_b \leq \bar{v}_{bj}\alpha_{bj} + (1 - \alpha_{bj})\bar{v}_b, \tag{31}$$

and for any $i \in \mathcal{J}_b$ s.t. $\mathcal{J}_b^+(i) \neq \emptyset$,

$$\sum_{j\in\mathcal{J}_b^+(i)} \alpha_{bj} = \alpha_{bi}. \tag{32}$$

Moreover we add the following constraint for every $j \in \mathcal{J}_b$,

$$R_{bb} + \underline{v}_{bj}\bar{v}_{bj} \leq (\underline{v}_{bj} + \bar{v}_{bj})L_b + (\bar{v}_b^2 + \underline{v}_{bj}\bar{v}_{bj})(1 - \alpha_{bj}). \tag{33}$$

For every $j \in \mathcal{J}_b$ and for all $a \in \mathcal{B}$ s.t. $(b, a) \in \mathcal{E}$, we add the inequalities

$$R_{ba} \geq \underline{v}_{bj}L_a + \underline{v}_a L_b - \underline{v}_{bj}\underline{v}_a + \bar{v}_b\bar{v}_a(\alpha_{bj} - 1) \qquad R_{ba} \geq \bar{v}_{bj}L_a + \bar{v}_a L_b - \bar{v}_{bj}\bar{v}_a + \bar{v}_b\bar{v}_a(\alpha_{bj} - 1) \tag{34}$$

$$R_{ba} \leq \bar{v}_{bj}L_a + \underline{v}_a L_b - \underline{v}_a\bar{v}_{bj} + \bar{v}_b\bar{v}_a(1 - \alpha_{bj}) \qquad R_{ba} \leq \bar{v}_a L_b + \underline{v}_{bj}L_a - \bar{v}_a\underline{v}_{bj} + \bar{v}_b\bar{v}_a(1 - \alpha_{bj}). \tag{35}$$

4.2.2 Partitioning Phase Angle Difference intervals

For any $(b, a) \in \mathcal{E}$, we may want to split the interval $[\underline{\theta}_{ba}, \bar{\theta}_{ba}]$ in subintervals. We introduce a tree $\mathcal{J}_{ba}$ and pairs $(\underline{\theta}_{baj}, \bar{\theta}_{baj})$ s.t. $\underline{\theta}_{baj} \leq \bar{\theta}_{baj}$ for all $j \in \mathcal{J}_{ba}$. For r being the root node of $\mathcal{J}_{ba}$, we have $(\underline{\theta}_{bar}, \bar{\theta}_{bar}) = (\underline{\theta}_{ba}, \bar{\theta}_{ba})$. Denoting $\mathcal{J}_{ba}^+(i)$ the set of child nodes of i, the partition $[\underline{\theta}_{bai}, \bar{\theta}_{bai}] = \bigcup_{j\in\mathcal{J}_{ba}^+(i)} [\underline{\theta}_{baj}, \bar{\theta}_{baj}]$ holds for any $i \in \mathcal{J}_{ba}$. For $j \in \mathcal{J}_{ba}$, we introduce a variable $\beta_{baj} \in \{0, 1\}$. To encode the equivalence $(\beta_{baj} = 1) \iff (\angle W_{ba} \in [\underline{\theta}_{baj}, \bar{\theta}_{baj}])$, we impose $\beta_{bar} = 1$ and for any $j \in \mathcal{J}_{ba}$

$$\tan(\underline{\theta}_{ba})\mathsf{Re}(W_{ba}) + (\beta_{baj} - 1)\bar{v}_b\bar{v}_a \leq \mathsf{Im}(W_{ba}) \leq \tan(\bar{\theta}_{ba})\mathsf{Re}(W_{ba}) + (1 - \beta_{baj})\bar{v}_b\bar{v}_a, \tag{36}$$

and for any $i \in \mathcal{J}_{ba}$ s.t. $\mathcal{J}_{ba}^+(i) \neq \emptyset$

$$\sum_{j\in\mathcal{J}_{ba}^+(i)} \beta_{baj} = \beta_{bai}. \tag{37}$$

Moreover, for all $j \in \mathcal{J}_{ba}$, we define the angles $\phi_{baj} = \frac{\underline{\theta}_{baj}+\bar{\theta}_{baj}}{2}$ and $\delta_{baj} = \frac{\bar{\theta}_{baj}-\underline{\theta}_{baj}}{2}$, and if $\delta_{baj} \leq \frac{\pi}{2}$, we impose

$$\cos(\phi_{baj})\mathsf{Re}(W_{ba}) + \sin(\phi_{baj})\mathsf{Im}(W_{ba}) \geq R_{ba}\cos(\delta_{baj}) + (\beta_{baj} - 1)\bar{v}_b\bar{v}_a. \tag{38}$$

4.3 Updating the partitions of the intervals

During the algorithm presented in Section 4.4, the partitions of the intervals $[\underline{v}_b, \bar{v}_b]$ and $[\underline{\theta}_{ba}, \bar{\theta}_{ba}]$ are not static but are made dynamically. The partition trees are all initialized as single-node graphs, and are then updated over the course of the algorithm. We present how these trees are updated, at any iteration t of the algorithm, where the current iterate is (S^t, W^t, L^t, R^t).

For a given bus $b \in \mathcal{B}$, we update the partition tree $\mathcal{J}_b$ by selecting the active leaf j, i.e. the only leaf j of $\mathcal{J}_b$ s.t. $L_b^t \in [\underline{v}_{bj}, \bar{v}_{bj}]$. We create three new leaves j_1, j_2, j_3 in the tree, which are attached to node j, and we partition the interval $[\underline{v}_{bj}, \bar{v}_{bj}]$ as follows:

- We define $\underline{v}_{bj_1} = \underline{v}_{bj}$ and $\bar{v}_{bj_3} = \bar{v}_{bj}$;
- If $L_b^t \leq \frac{\underline{v}_{bj}+\bar{v}_{bj}}{2}$, we define $\bar{v}_{bj_1} = \underline{v}_{bj_2} = L_b^t$ and $\bar{v}_{bj_2} = \underline{v}_{bj_3} = \frac{L_b^t+\bar{v}_{bj}}{2}$;
- Else, we define $\bar{v}_{bj_1} = \underline{v}_{bj_2} = \frac{\underline{v}_{bj}+L_b^t}{2}$ and $\bar{v}_{bj_2} = \underline{v}_{bj_3} = L_b^t$.

For a given pair $(b, a) \in \mathcal{E}$, we update the partition tree $\mathcal{J}_{ba}$ by selecting the active leaf j, i.e. the only leaf j of $\mathcal{J}_{ba}$ s.t. $\angle W_{ba}^t \in [\underline{\theta}_{baj}, \bar{\theta}_{baj}]$. We create three new leaves j_1, j_2, j_3 in the tree, which are attached to node j, and we partition the interval $[\underline{\theta}_{baj}, \bar{\theta}_{baj}]$ as follows:

- We define $\underline{\theta}_{baj_1} = \underline{\theta}_{baj}$ and $\bar{\theta}_{baj_3} = \bar{\theta}_{baj}$;
- If $\angle W_{ba}^t \leq \frac{\underline{\theta}_{baj}+\bar{\theta}_{baj}}{2}$, we define $\bar{\theta}_{baj_1} = \underline{\theta}_{baj_2} = \angle W_{ba}^t$ and $\bar{\theta}_{baj_2} = \underline{\theta}_{baj_3} = \frac{\angle W_{ba}^t+\bar{\theta}_{baj}}{2}$;
- Else, we define $\bar{\theta}_{baj_1} = \underline{\theta}_{baj_2} = \frac{\underline{\theta}_{baj}+\angle W_{ba}^t}{2}$ and $\bar{\theta}_{baj_2} = \underline{\theta}_{baj_3} = \angle W_{ba}^t$.

The construction procedure of the trees $\mathcal{J}_b$ and $\mathcal{J}_{ba}$ guarantees that (i) $\bar{v}_{bj} - \underline{v}_{bj}$, the length of the interval associated with a node $j \in \mathcal{J}_b$ of depth $D(j)$, is less than $\frac{\Delta_b}{2^{D(j)}}$, (ii) the coefficient δ_{baj} associated with a node $j \in \mathcal{J}_{ba}$ is less than $\frac{\delta_{ba}}{2^{D(j)}}$.

▶ **Proposition 7.** *We assume that the convex Constraints* (8)–(13) *and the MILP Constraints* (31)–(38) *are satisfied, but with a tolerance $\rho \in [0,1]$ for the nonlinear Constraints* (10) *and* (12)*. Then, for any nodes $j_b \in \mathcal{J}_b, j_a \in \mathcal{J}_a$ and $j_{ba} \in \mathcal{J}_{ba}$ that are active, i.e., s.t. $\alpha_{bj_b} = \alpha_{aj_a} = \beta_{baj_{ba}} = 1$, we have*

$$\left|(R_{ba})^2 - W_{bb}W_{aa}\right| \leq \frac{9\Delta_b\Delta_a}{2^{\max\{D(j_b),D(j_a)\}}} + \max\left\{9\rho, \left(\frac{\Delta_b}{2^{D(j_b)}} + \frac{\Delta_a}{2^{D(j_a)}}\right)^2\right\}, \tag{39}$$

$$\left(D(j_{ba}) \geq \log_2\left(\frac{2\delta_{ba}}{\pi}\right)\right) \implies \left(\left||W_{ba}|^2 - (R_{ba})^2\right| \leq \max\left\{9\rho, \frac{16\delta_{ba}^2}{4^{D(j_{ba})}}\right\}\right). \tag{40}$$

We underline that the implication is still valid if $\delta_{ba} = 0$ and $\log_2(\frac{2\delta_{ba}}{\pi}) = -\infty$.

Proof. Since $\alpha_{bj_b} = 1$, Constraints (34)–(35) yield Constraints (8)–(9), but with $\underline{v}_b$, $\bar{v}_b$ and Δ_b replaced by $\underline{v}_{bj_b}$, $\bar{v}_{bj_b}$ and $\widetilde{\Delta}_b = \bar{v}_{bj_b} - \underline{v}_{bj_b} \leq \frac{\Delta_b}{2^{D(j_b)}}$. Applying the first point of Theorem 4 with these parameters, we deduce that $|(R_{ba})^2 - L_a^2 L_b^2| \leq 9\widetilde{\Delta}_b\Delta_a \leq 9\frac{\Delta_b}{2^{D(j_b)}}\Delta_a$. Similarly, since $\alpha_{bj_a} = 1$ and since $R_{ba} = R_{ab}$, we also deduce from Constraints (34)–(35) that $|(R_{ba})^2 - L_a^2 L_b^2| \leq 9\frac{\Delta_a}{2^{D(j_a)}}\Delta_b$. Hence, we obtain

$$|(R_{ba})^2 - L_a^2 L_b^2| \leq \frac{9\Delta_b\Delta_a}{2^{\max\{D(j_b),D(j_a)\}}}. \tag{41}$$

Since $\alpha_{bj_b} = 1$ (resp. $\alpha_{aj_a} = 1$), Constraint (33) yields Constraint (11) for b (resp. a) with $\underline{v}_b$, $\bar{v}_b$ and Δ_b (resp. $\underline{v}_a$, $\bar{v}_a$ and Δ_a) replaced by $\underline{v}_{bj_b}$, $\bar{v}_{bj_b}$ and $\widetilde{\Delta}_b$ (resp. $\underline{v}_{aj_a}$, $\bar{v}_{bj_a}$ and $\widetilde{\Delta}_a$). Applying Equation (16) in the Proof of Theorem 4, that follows only from Constraint (11), we deduce that $R_{bb}R_{aa} - L_b^2 L_a^2 \leq (\widetilde{\Delta_b} + \widetilde{\Delta_a})^2 \leq (\frac{\Delta_b}{2^{D(j_b)}} + \frac{\Delta_a}{2^{D(j_a)}})^2$. Since Constraint (10) is satisfied with tolerance $\rho \in [0,1]$, we have that $L_b^2 \leq R_{bb} + \rho$ and $L_a^2 \leq R_{aa} + \rho$. Multiplying both inequalities, we deduce that $L_b^2 L_a^2 \leq R_{bb}R_{aa} + \rho(R_{bb} + R_{aa}) + \rho^2 \leq R_{bb}R_{aa} + 9\rho$, since $R_{bb}, R_{aa} \in [0,4]$ and $\rho^2 \leq \rho$. Hence, $|W_{bb}W_{aa} - L_b^2 L_a^2| = |R_{bb}R_{aa} - L_b^2 L_a^2| \leq \max\{9\rho, (\frac{\Delta_b}{2^{D(j_b)}} + \frac{\Delta_a}{2^{D(j_a)}})^2\}$. Combining this with Equation (41), we deduce Equation (39) due to the triangle inequality. We assume now that $D(j_{ba}) \geq \log_2(\frac{2\delta_{ba}}{\pi})$, implying that $\delta_{baj_{ba}} \leq \frac{\delta_{ba}}{2^{D(j_{ba})}} \leq \frac{\pi}{2}$. Since $\beta_{baj_{ba}} = 1$, Constraint (38) yields Constraint (13) with ϕ_{ba}, δ_{ba} replaced by $\phi_{baj_{ba}}, \delta_{baj_{ba}}$. Writing W_{ba} as $|W_{ba}|e^{i\theta}$ we thus have $|W_{ba}|(\cos(\phi_{baj_{ba}})\cos(\theta) + \sin(\phi_{baj_{ba}})\sin(\theta)) \geq R_{ba}\cos(\delta_{baj_{ba}})$. This also reads $|W_{ba}|\cos(\phi_{baj_{ba}} - \theta) \geq R_{ba}\cos(\delta_{baj_{ba}})$. This implies that $|W_{ba}| \geq R_{ba}\cos(\delta_{baj_{ba}})$, and thus $|W_{ba}|^2 \geq R_{ba}^2\cos(\delta_{baj_{ba}})^2$. Noticing that Constraint (12) is satisfied with tolerance ρ, we have that $|W_{ba}| \leq R_{ba} + \rho$ and $|W_{ba}|^2 \leq R_{ba}^2 + 2R_{ba}\rho + \rho^2 \leq R_{ba}^2 + 9\rho$. In summary, we have $-9\rho \leq R_{ba}^2 - |W_{ba}|^2 \leq R_{ba}^2(1 - \cos(\delta_{baj_{ba}})^2) \leq R_{ba}^2\sin(\delta_{baj_{ba}})^2 \leq 16(\delta_{baj_{ba}})^2 \leq 16(\frac{\delta_{ba}}{2^{D(j_{ba})}})^2$. ◀

4.4 The MILP-based iterative scheme

The following Global Optimization algorithm is executed based on (i) a local NLP solver (ii) a Conic Programming solver and (iii) a MILP solver. In this pseudo-code, we use the function $\epsilon(W) = \max_{(b,a)\in\mathcal{E}} \big||W_{ba}|^2 - W_{bb}W_{aa}\big|$, which denotes the feasibility error in Constraints ($\star$).

0. Input: A target optimality gap $\mathsf{targetOptGap} \geq 0$, a tolerance $\bar{\epsilon} \geq 0$, integers $N_1, N_2 \in \mathbb{N}^*$ and a sequence $(\rho_t)_{t\in\mathbb{N}}$ with $\rho_t > 0$.

1. Initialization: Compute an ACOPF feasible solution with a NLP solver and denote by $\overline{\mathsf{obj}}$ its value (if the NLP solver fails, $\overline{\mathsf{obj}} \leftarrow +\infty$). Solve the Conic Programming relaxation ($\mathbf{R}$). If the gap is greater than $\mathsf{targetOptGap}$, apply FBBT and OBBT to ($\mathbf{R}$). Based on the optimal solution of Problem ($\mathbf{R}$), generate the LP relaxation ($\mathbf{R_L}$) with same value as ($\mathbf{R}$) (see Subsect. 4.1). Set $t \leftarrow 0$ and $\mathsf{LB}_t \leftarrow \mathsf{val}(\mathbf{R})$.

2. Outer-iterations: While (i) $\overline{\mathsf{obj}} - \mathsf{LB}_t > \mathsf{targetOptGap}$ and (ii) $\epsilon(W) > \bar{\epsilon}$, do:

2.1. For N_1 couples $(b,a) \in \mathcal{E}$ with largest violation $|R_{ba}^2 - W_{bb}W_{aa}|$, update the partition trees $\mathcal{J}_b$ and $\mathcal{J}_a$ according to Section 4.3 and add the corresponding Constraints (31)–(35) to the MILP relaxation.

2.2. For N_2 couples $(b,a) \in \mathcal{E}$ with largest violation $\big||W_{ba}|^2 - R_{ba}^2\big|$, update the partition tree $\mathcal{J}_{ba}$ according to Section 4.3 and add the corresponding Constraints (36)–(38) to the MILP relaxation.

2.3. Solve the resulting MILP relaxation to global optimality to get (S, W, L, R) and enter the inner loop (step 3.). After the end of the inner loop, set LB_{t+1} as the value of the MILP relaxation and set $t \leftarrow t+1$.

3. Inner-iterations: While $x = (\mathsf{Re}(S), \mathsf{Im}(S), \mathsf{Re}(W), \mathsf{Im}(W), L, R)$ does not satisfy the convex constraints within tolerance ρ_t, i.e., while $\max_j f_j(x) - \breve{f}_j(x) > \rho_t$,

3.1. Add the corresponding cuts: $\breve{\mathcal{U}}_j \leftarrow \breve{\mathcal{U}}_j \cup \{u\}$, for all $j \in \{0, \ldots, M\}$ and for $u \in \mathcal{U}_j$ s.t. $f_j(x) = u^\top \pi_j(x)$.

3.2. Solve the resulting MILP problem to global optimality to get $x = (\mathsf{Re}(S), \mathsf{Im}(S), \mathsf{Re}(W), \mathsf{Im}(W), L, R)$.

Theorem 9 states that, if the parameters $\mathsf{targetOptGap}$ and $\bar{\epsilon}$ are set to zero and if $(\rho_t)_{t\in\mathbb{N}}$ vanishes, the algorithm asymptotically recovers global minimizers of ($\mathbf{ACOPF_W}$). Before stating this Theorem, we introduce a preliminary Proposition about the finite termination of the inner-loops.

▶ **Proposition 8.** *For any outer-iteration index $t \in \mathbb{N}^*$, for any tolerance $\rho_t > 0$, the inner-loop has a finite number of iterations.*

Proof. During outer-iteration $t \in \mathbb{N}^*$ and the previous iterations, several auxiliary binary variables and associated linear constraints have been added to the relaxation ($\mathbf{R}$). From the perspective of the decision vector $x = (\mathsf{Re}(S), \mathsf{Im}(S), \mathsf{Re}(W), \mathsf{Im}(W), L, R)$, this yields a closed nonconvex set $\mathcal{X}$. We also inherit finite sets $(\breve{U}_{j0})_{j\in\{0,1,\ldots,M\}}$, the subscript 0 denoting the inner-iteration of index $s = 0$. The inner-iteration $s \in \mathbb{N}$ consists in solving

$$\begin{cases} \min_{x\in\mathcal{P}\cap\mathcal{X},\lambda\in\mathbb{R}} & \lambda \\ \forall\, u \in \breve{\mathcal{U}}_{0s} & u^\top \pi_0(x) \leq \lambda \\ \forall\, j \in \{1,\ldots,M\},\ \forall\, u \in \breve{\mathcal{U}}_{js} & u^\top \pi_j(x) \leq 0, \end{cases} \tag{42}$$

to obtain a solution x_s, and in defining $\mathcal{U}_{j(s+1)} = \{u_{js}\} \cup \mathcal{U}_{js}$ for some $u_{js} \in \mathrm{argmax}_{u\in\mathcal{U}_j} u^\top \pi_j(x_s)$ for all $j \in \{0,\ldots,M\}$. We define the error $e_{js} = f_j(x_s) - \breve{f}_{js}(x_s) = f_j(x_s) - \max_{u\in\breve{\mathcal{U}}_{js}} u^\top \pi_j(x_s)$. We reason by contrapositive and assume now that the inner-loop is not terminating in finite time, meaning that the generated MILP relaxation is feasible at each inner-iteration and the stopping condition of the inner-loop is never met. This second point means that $\rho_t \leq \max_{j\in\{0,\ldots,M\}} e_{js}$ for all $s \in \mathbb{N}$. We take any $j \in \{0,\ldots,M\}$ and show that $e_{js} \to_s 0$. Since the sets $\mathcal{P} \cap \mathcal{X}$ and $\mathcal{U}_j$ are compact and since the functions $x \mapsto f_j(x)$ and $(x,u) \mapsto u^\top \pi_j(x)$ are continuous, we deduce that the sequence $(e_{js})_{s\in\mathbb{N}}$ is bounded. We take any limit point e^* of this sequence, and we take a converging subsequence $(e_{j\psi(s)}) \to_s e^*$. Without loss of generality, since $\mathcal{P} \cap \mathcal{X}$ and $\mathcal{U}_j$ are compact, we can choose the extraction $\psi(s)$ so that $x_{\psi(s)} \to_s x^*$ and $u_{j\psi(s)} \to_s u^*$, for $(x^*, u^*) \in \mathcal{P} \cap \mathcal{X} \times \mathcal{U}_j$. For $s \in \mathbb{N}^*$, we notice that

$$e_{j\psi(s)} = f_j(x_{\psi(s)}) - \max_{u\in\mathcal{U}_{j\psi(s)}} u^\top \pi_j(x_{\psi(s)}) \tag{43}$$

$$\leq f_j(x_{\psi(s)}) - u_{j\psi(s-1)}^\top \pi_j(x_{\psi(s)}) \tag{44}$$

$$\leq u_{j\psi(s)}^\top \pi_j(x_{\psi(s)}) - u_{j\psi(s-1)}^\top \pi_j(x_{\psi(s)}). \tag{45}$$

Indeed, Equation (44) follows from the fact that $u_{j\psi(s-1)} \in \mathcal{U}_{j\psi(s)}$ since $\psi(s-1) \leq \psi(s) - 1$, and Equation (45) follows from the definition of $u_{j\psi(s)}$. The limit of the term in Equation (45) is $u^*\pi_j(x^*) - u^*\pi_j(x^*) = 0$.

Since $0 \le e_{j\psi(s)}$, this proves that $(e_{j\psi(s)}) \to_s 0$, i.e., $e^* = 0$. This being true for any limit value e^* of the bounded sequence (e_{js}), we deduce that it converges to zero. This implies that $(\max_{j\in\{0,\dots,M\}} e_{js}) \to_s 0$. As $\rho_t \le \max_{j\in\{0,\dots,M\}} e_{js}$, we see that the hypothesis "the inner-loop is not terminating" implies that $\rho_t = 0$. By contrapositive, since $\rho_t > 0$, we deduce that the inner-loop has a finite number of iterations. ◀

▶ **Theorem 9.** *If* targetOptGap $= \bar{\epsilon} = 0$ *and* $\rho_t \to_t 0$, *then*

- *Either the algorithm stops due to the stopping criterion, and yields a global minimizer of* $(\mathbf{ACOPF}_W)$,
- *Or the algorithm stops due to the infeasibility of a relaxation, certifying the infeasibility of* $(\mathbf{ACOPF}_W)$,
- *Or the algorithm generates an infinite sequence of iterates* (S^t, W^t, L^t, R^t), *and*
  - *The sequence* LB_t *monotonously converges to* $\mathsf{val}(\mathbf{ACOPF_W}) = \mathsf{val}(\mathbf{ACOPF})$,
  - *The limit points of the sequence* $(S^t, W^t)_{t\in\mathbb{N}}$ *are global minimizers of* $(\mathbf{ACOPF_W})$.

Proof. We consider the first case where the algorithm meets the stopping criterion at the beginning of a certain outer-iteration t. This means that either (a) $\overline{\mathsf{obj}} = \mathsf{LB}_t$, proving that the solution (S, V) found by the NLP solver at step 1. is globally optimal in $(\mathbf{ACOPF})$ and yields (S, VV^{H}) globally optimal in $(\mathbf{ACOPF}_W)$, or (b) the solution (S^t, W^t, L^t, R^t) of the current MILP relaxation of $(\mathbf{ACOPF}_W)$ satisfies $\epsilon(W^t) = 0$, i.e., (S^t, W^t) is in fact feasible in $(\mathbf{ACOPF}_W)$ and thus optimal in $(\mathbf{ACOPF}_W)$ since it is the optimal solution of a relaxation.

The second case is trivial: if the relaxation $(\mathbf{R})$ or any MILP relaxation during the iterations is infeasible, this implies that $(\mathbf{ACOPF}_W)$ is also infeasible.

We consider now the third case, where the algorithm does not terminate. We invoke Proposition 8 to claim that for any outer-iteration $t \in \mathbb{N}$, the inner-loop terminates in finite time. Hence, there is an infinite number of outer-iterations and we define the infinite sequence $x_t = (S^t, W^t, L^t, R^t)_{t\in\mathbb{N}}$, where x_t is the solution of the MILP relaxation at the beginning of the outer-iteration t. For any $(b, a) \in \mathcal{E}$, we define $\chi^t_{ba} = |(R^t_{ba})^2 - W^t_{bb}W^t_{aa}|$ and $\xi^t_{ba} = \big||W^t_{ba}|^2 - (R^t_{ba})^2\big|$. We let $\mathcal{J}^t_b$ (resp. $\mathcal{J}^t_{ba}$) denote the state of the tree $\mathcal{J}_b$ (resp. $\mathcal{J}_{ba}$) at the beginning of iteration t, and $\overline{\mathcal{J}}_b$ (resp. $\overline{\mathcal{J}}_{ba}$) the (potentially infinite) limit tree $\bigcup_t \mathcal{J}^t_b$ (resp. $\bigcup_t \mathcal{J}^t_{ba}$). We first show that $\chi^t_{ba} \to_t 0$ for any $(b, a) \in \mathcal{E}$. For $t \in \mathbb{N}$, we define $(b_t, a_t) \in \mathcal{E}$ s.t. $\chi^t_{b_ta_t} = \max_{ba} \chi^t_{ba}$, and we define $j_b(t) \in \mathcal{J}^t_{b_t}$ and $j_a(t) \in \mathcal{J}^t_{a_t}$ the active leaves to which three child nodes are attached during step 2.1 since (b_t, a_t) presents the largest violation. For any $j \in \bigcup_b \overline{\mathcal{J}}_b$, we recall that $D(j)$ is the depth of j in the (unique) tree $\overline{\mathcal{J}}_b$ it belongs to. As x_t is the output of the outer-iteration $t-1$, it satisfies Constraint (10) and (12) with tolerance ρ_{t-1}, and we can apply Proposition 7 with $\rho = \rho_{t-1}$. This yields

$$\chi^t_{b_ta_t} = \big|(R^t_{b_ta_t})^2 - W^t_{b_tb_t}W^t_{a_ta_t}\big| \le \frac{9\Delta_{b_t}\Delta_{a_t}}{2^{\max\{D(j_b(t)),D(j_a(t))\}}} + \max\left\{9\rho_{t-1}, \left(\frac{\Delta_{b_t}}{2^{D(j_b(t))}} + \frac{\Delta_{a_t}}{2^{D(j_a(t))}}\right)^2\right\}. \tag{46}$$

We notice that the sequence $(j_b(t))_{t\in\mathbb{N}}$ is injective, since each $j_b(t)$ is a leaf in $\mathcal{J}^t_{b_t}$, but not in the trees $\mathcal{J}^s_{b_t}$ for $s \ge t+1$. We deduce that $D(j_b(t)) \to_t \infty$, otherwise by contrapositive, there would exist $M \in \mathbb{N}$ s.t. an infinite number of nodes of depth less or equal than M are created in the union of ternary trees $\bigcup_b \overline{\mathcal{J}}_b$; This is false since the number of nodes with depth less or equal than M is bounded by $n\sum_{\ell=0}^M 3^\ell$. By the same argument, we have $D(j_a(t)) \to_t \infty$. Combined with (46), we deduce that $\chi^t_{b_ta_t} \to_t 0$ since $\rho_t \to_t 0$ and because $\Delta_{b_t}, \Delta_{a_t}$ are bounded. For any $t \in \mathbb{N}$ and $(b, a) \in \mathcal{E}$, we have $0 \le \chi^t_{ba} \le \chi^t_{b_ta_t}$ by definition of (b_t, a_t), implying $\chi^t_{ba} \to_t 0$.

We apply the same approach to prove that $\xi^t_{ba} \to_t 0$ for any $(b, a) \in \mathcal{E}$. For $t \in \mathbb{N}$, we define $(\tilde{b}_t, \tilde{a}_t) \in \mathcal{E}$ s.t. $\xi^t_{\tilde{b}_t\tilde{a}_t} = \max_{ba} \xi^t_{ba}$, and we define $\tilde{j}(t) \in \mathcal{J}^t_{\tilde{b}_t\tilde{a}_t}$ the active leaf to which three child nodes are attached during step 2.2. We also define $D(\tilde{j}(t))$ as the depth of $\tilde{j}(t)$ in $\mathcal{J}^t_{\tilde{b}_t\tilde{a}_t}$, which satisfies $D(\tilde{j}(t)) \to_t \infty$ by injectivity of $(\tilde{j}(t))_{n\in\mathbb{N}}$ and since the number of nodes in $\bigcup_{(b,a)\in\mathcal{E}} \overline{\mathcal{J}}_{ba}$ with depth less or equal than M is bounded by $|\mathcal{E}|\sum_{\ell=0}^M 3^\ell$. As $D(\tilde{j}(t)) \to_t \infty$, we know that it exists $t_0 \in \mathbb{N}$ s.t. $D(\tilde{j}(t)) \ge 2$ for all $t \ge t_0$. Hence, for all $t \ge 0$, $D(\tilde{j}(t)) \ge \log_2(\frac{4\pi}{\pi}) \ge \log_2(\frac{2\delta_{\tilde{b}_t\tilde{a}_t}}{\pi})$, since $\delta_{b_ta_t} \in [0, 2\pi]$. Applying Proposition 7, we know that for any $t \ge t_0$,

$$\xi^t_{\tilde{b}_t\tilde{a}_t} = \big||W^t_{\tilde{b}_t\tilde{a}_t}|^2 - (R^t_{\tilde{b}_t\tilde{a}_t})^2| \le \max\left\{9\rho_{t-1}, \frac{16(\delta_{\tilde{b}_t\tilde{a}_t})^2}{4^{D(\tilde{j}(t))}}\right\} \le \max\left\{9\rho_{t-1}, \frac{64\pi^2}{4^{D(\tilde{j}(t))}}\right\}, \tag{47}$$

Combined with $\rho_t \to_t 0$ and $D(\tilde{j}(t)) \to_t \infty$, we deduce that $\xi^t_{\tilde{b}_t\tilde{a}_t} \to_t 0$. Additionally, since $\xi^t_{\tilde{b}_t\tilde{a}_t} = \max_{ba} \xi^t_{ba}$, we have $0 \le \xi^t_{ba} \le \xi^t_{\tilde{b}_t\tilde{a}_t}$ and, thus, $\xi^t_{ba} \to_t 0$ for any $(b, a) \in \mathcal{E}$.

We deduce that $\epsilon(W^t) \to_t 0$, since $\epsilon(W^t) = \max_{(b,a)\in\mathcal{E}} bigl||W^t_{ba}|^2 - W^t_{bb}W^t_{aa}\big| \le \sum_{(b,a)\in\mathcal{E}} \big||W^t_{ba}|^2 - W^t_{bb}W^t_{aa}\big| \le \sum_{(b,a)\in\mathcal{E}} \chi^t_{ba} + \xi^t_{ba}$, due to the triangle inequality. Hence, for any limit point $(\overline{S}, \overline{W})$ of (S^t, W^t), we thus have $\epsilon(\overline{W}) = 0$. As $\rho_t \to_t 0$, this also proves that $(\overline{S}, \overline{W})$ satisfies the nonlinear convex constraints in $(\mathbf{R})$. Hence, $(\overline{S}, \overline{W})$

is feasible in ($\mathbf{ACOPF}_W$). We denote by $\breve{f}_0^t$ the cutting-plane model of the objective function at the beginning of iteration t; this function only depends on S, hence, we write $\breve{f}_0^t(S)$ instead of $\breve{f}_0^t(x)$. As the successive MILP relaxations over the iterations have nonincreasing feasible sets w.r.t. variables (S, W, L, R) and have nondecreasing sequence $\breve{f}_0^t(S)$ as objective functions, the sequence $\breve{f}_0^t(S^t) = \mathsf{LB}_t$ is nondecreasing. It is also bounded above by $\mathsf{val}(\mathbf{ACOPF}_W)$ and, thus, converges to a value $v^* \leq \mathsf{val}(\mathbf{ACOPF}_W)$. Since $\rho_t \to_t 0$, $\breve{f}_0^t(S^t) \to_t f_0(\overline{S}) = \sum_{g\in\mathcal{G}} c_{1g}\,\mathsf{Re}(\overline{S}_g) + c_{2g}\,\mathsf{Re}(\overline{S}_g)^2$, for any limit point $(\overline{S}, \overline{W})$. By uniqueness of the limit of $\breve{f}_0^t(S^t)$ and since $(\overline{S}, \overline{W})$ is feasible in ($\mathbf{ACOPF}_W$), we deduce that $v^* = \sum_{g\in\mathcal{G}} \left(c_{1g}\,\mathsf{Re}(\overline{S}_g) + c_{2g}\,\mathsf{Re}(\overline{S}_g)^2\right) \geq \mathsf{val}(\mathbf{ACOPF}_W)$. We conclude that $v^* = \mathsf{val}(\mathbf{ACOPF}_W) = \mathsf{val}(\mathbf{ACOPF})$ and that $(\overline{S}, \overline{W})$ is optimal in ($\mathbf{ACOPF}$). ◀

5 Experimental evaluation

5.1 Experimental setting

For all experiments, we use a 64-bit Ubuntu computer with 32 Intel(R) Xeon(R) CPU E5-2620 v4 @ 2.10GHz and 64 GB RAM. Along our algorithm, we use the commercial solvers MOSEK [1] and CPLEX [18] called through their Python APIs, as well as the academic solver IPOPT [33] called through the Pyomo interface [16]. We compute the tree decomposition with the approximate minimum degree (AMD) ordering routine of the `chompack` package. We consider a relative optimality gap of 0.01% for global optimality (**GOPT**) and use the parameters $(N_1, N_2, \bar{\epsilon}) = (4, 4, 10^{-6})$. The FBBT and OBBT procedures are applied for all variables a maximum of 4 times, and with a time limit of 10 hours (TL_1). After each pass of FBBT and OBBT, we apply the Floyd–Warshall algorithm and we check whether the gap of the tightened conic relaxation reaches the target optimality gap. If the maximum number of Bound Tightening iterations or time limit is reached, we enter the MILP iterative scheme with a time limit of 5 hours (TL_2). Our code is available at `github.com/aoustry/SDP-MILP4OPF`.

This study focuses on the network instances from the IEEE PES PGLib AC-OPF v21.07 library [2] with less than 500 buses. As shown in Table 2, the instances of this benchmark are split in three categories depending on their characteristics: Typical Operating Condition (TYP) instances correspond to a reference scenario, Congested Operating Condition (API) correspond to situations with greater Power Demands, and Small Angle Condition (SAD) correspond to tighter constraints for the Phase Angle Difference.

We compare our approach with the standard SOCP and SDP relaxations [28], and with two other Global Optimization approaches [13, 32]. We performed these comparative experiments on the same PGLib v21.07 instances, with the same hardware, the same time limit (TL_1) as our OBBT algorithm, and the same relative optimality gap tolerance. The concurrent approach from [32] is an OBBT algorithm based on a strengthened QC relaxation. We ran the Julia implementation of this algorithm provided in the PowerModels.jl package [8]. The QC relaxations are solved with IPOPT. The concurrent approach from [13] consists of an OBBT algorithm, based on a Determinant SDP relaxation strengthened with RLT constraints. We ran a C++ implementation of this algorithm, that is based on the Mathematical Modeling Language Gravity [17], and is available at the link indicated in [13]; The corresponding relaxations are also solved with IPOPT. We point out that these competing approaches [13, 32] solely rely on open-source tools, whereas our implementation uses commercial solvers.

5.2 Numerical results

Table 2 presents the optimality gap (in %) and the computational times obtained by the several approaches for the considered list of instances. For lack of space, we do not detail the computation time of the SOCP and SDP relaxations. To give an idea, the computation time of the SOCP relaxation is below 2s for all instances; for the SDP relaxation, the computation time goes from 0.2s for the smallest cases to about 40s for the largest cases. As regards the column "This work", in the optimality gap section of the table: the entry (**GOPT**) means that the Bound Tightening procedure based on the Conic Programming relaxation closed the gap; else, the entry a/b represents the gap after the Bound Tightening procedure (a) and the gap after the iterative MILP scheme (b).

This table shows that our algorithm reaches Global Optimality for 35 instances over 51. The optimality gap is below 0.5% for 43 instances over 51. For 4 instances only, the optimality gap at the end of the algorithm is above 2%. As regards the instances with less than 57 buses, they are all solved to Global Optimality in less than 220 seconds. For all these instances with less than 57 buses, except case5_pjm and case30_as_api, the Bound Tightening procedure based on the Conic Programming relaxation (**R**) manages to close the gap. For the instances case5_pjm and case30_as_api, the gap is closed by the iterative MILP scheme. For all the instances with more than 57 nodes where the MILP scheme is executed, the gap is admittedly not closed within the time limit, but it is reduced, except for case500_goc_api. For 44 over these 51 instances, our algorithm has the lowest gap; for 14 instances over 51, it has a strictly lower gap than the others approaches. For 5 instances (among those 14 instances), our approach is the only one to reach Global Optimality. Regarding the 7 instances where our approach has not the best gap: for 3 instances, only the QC relaxation-based Bound Tightening algorithm [32] yields a strictly lower gap than our approach; for the 4 other instances, only the Determinant-SDP relaxation-based Bound Tightening algorithm [13] yields a strictly lower gap than our approach.

Table 2 Results for the instances from IEEE PES PGLib AC-OPF v21.07 with less than 500 buses

	Optimality gap (%)					Time (s)			
		Benchmark			This work	Benchmark		This work	
Case	SOCP	SDP	[32]	[13]	(**R**)-BT/MILP	[32]	[13]	(**R**)-BT	MILP
Typical Operating Condition (TYP)									
case3_lmbd	1.32	0.39	**GOPT**	**GOPT**	**GOPT**	1	<1	1	0
case5_pjm	14.55	5.21	5.76	**GOPT**	*5.01*/**GOPT**	44	30	7	198
case14_ieee	0.11	**GOPT**	**GOPT**	**GOPT**	**GOPT**	12	<1	3	0
case24_ieee_rts	0.02	**GOPT**	**GOPT**	**GOPT**	**GOPT**	41	10	7	0
case30_as	0.06	**GOPT**	**GOPT**	**GOPT**	**GOPT**	107	5	10	0
case30_ieee	18.84	**GOPT**	**GOPT**	**GOPT**	**GOPT**	226	3	8	0
case39_epri	0.56	**GOPT**	**GOPT**	**GOPT**	**GOPT**	272	2	9	0
case57_ieee	0.16	**GOPT**	**GOPT**	**GOPT**	**GOPT**	437	280	13	0
case73_ieee_rts	0.04	**GOPT**	**GOPT**	**GOPT**	**GOPT**	399	30	21	0
case89_pegase	0.75	0.37	0.32	**0.08**	*0.27*/0.18	TL_1	TL_1	16,582	TL_2
case118_ieee	0.91	0.07	**GOPT**	**GOPT**	**GOPT**	8,527	TL_1	783	0
case162_ieee_dtc	5.95	1.78	**0.04**	1.58	*0.59*/0.53	TL_1	TL_1	TL_1	TL_2
case179_goc	0.16	0.07	0.04	0.05	*0.04*/**0.03**	TL_1	TL_1	5,293	TL_2
case200_activ	0.01	**GOPT**	**GOPT**	**GOPT**	**GOPT**	1,998	1,573	54	0
case240_pserc	2.78	1.43	2.71	1.21	*1.02*/**0.93**	TL_1	TL_1	25,578	TL_2
case300_ieee	2.63	1.03	2.55	0.05	**GOPT**	TL_2	TL_1	5,482	0
case500_goc	0.25	**GOPT**	0.19	**GOPT**	**GOPT**	TL_1	10,040	139	0
Congested Operating Condition (API)									
case3_lmbd_api	9.27	7.10	**GOPT**	**GOPT**	**GOPT**	2	3	2	0
case5_pjm_api	4.09	0.26	**GOPT**	**GOPT**	**GOPT**	10	51	3	0
case14_ieee_api	5.13	**GOPT**	**GOPT**	**GOPT**	**GOPT**	55	1	4	0
case24_ieee_rts_api	17.9	2.07	**GOPT**	**GOPT**	**GOPT**	1,311	67	97	0
case30_as_api	44.6	10.86	36.1	0.71	*0.13*/**GOPT**	3,331	1,526	216	2
case30_ieee_api	5.46	**GOPT**	**GOPT**	0.02	**GOPT**	444	1,673	8	0
case39_epri_api	1.72	0.20	**GOPT**	**GOPT**	**GOPT**	622	1,228	96	0
case57_ieee_api	0.08	**GOPT**	**GOPT**	**GOPT**	**GOPT**	434	5	13	0
case73_ieee_rts_api	12.9	2.90	0.29	0.23	*0.28*/**0.08**	TL_1	TL_1	2,096	TL_2
case89_pegase_api	23.1	22.0	18.3	**17.6**	*21.7*/19.3	TL_1	TL_1	16,085	TL_2
case118_ieee_api	30.0	11.7	3.10	1.44	*1.26*/**0.90**	21,152	TL_1	3,491	TL_2
case162_ieee_dtc_api	4.36	1.44	0.27	1.31	*0.29*/**0.25**	TL_1	TL_1	TL_1	TL_2
case179_goc_api	9.88	0.59	0.54	**0.39**	*0.54*/0.53	TL_1	TL_1	5,992	TL_2
case200_activ_api	0.03	1.49	**GOPT**	**GOPT**	**GOPT**	2,561	1,472	55	0
case240_pserc_api	0.67	0.28	0.62	$\star$[1]	*0.12*/**0.11**	TL_1	$\star$	26,401	TL_2
case300_ieee_api	0.85	0.09	0.81	0.07	**GOPT**	TL_1	TL_1	7,841	0
case500_goc_api	3.44	2.36	3.28	**2.12**	*2.19*/2.19	TL_1	TL_1	TL_1	TL_2
Small Angle Difference (SAD)									
case3_lmbd_sad	3.75	1.86	**GOPT**	**GOPT**	**GOPT**	1	2	1	0
case5_pjm_sad	3.62	**GOPT**	**GOPT**	**GOPT**	**GOPT**	4	<1	1	0
case14_ieee_sad	21.53	0.09	**GOPT**	0.11	**GOPT**	40	367	17	0
case24_ieee_rts_sad	9.55	4.35	**GOPT**	**GOPT**	**GOPT**	603	1,050	98	0
case30_as_sad	7.88	0.24	**GOPT**	0.10	**GOPT**	168	478	71	0
case30_ieee_sad	9.70	**GOPT**	**GOPT**	**GOPT**	**GOPT**	161	5	9	0
case39_epri_sad	0.67	0.02	**GOPT**	**GOPT**	**GOPT**	193	1,101	90	0
case57_ieee_sad	0.71	0.05	**GOPT**	**GOPT**	**GOPT**	661	4,925	212	0
case73_ieee_rts_sad	6.73	2.74	**GOPT**	0.06	**GOPT**	8,818	TL_1	1,472	0
case89_pegase_sad	0.73	0.37	0.33	0.29	*0.28*/**0.19**	TL_1	TL_1	16,590	TL_2
case118_ieee_sad	8.17	3.25	0.02	0.27	**GOPT**	TL_1	TL_1	1,693	0
case162_ieee_dtc_sad	6.48	2.07	**0.02**	1.35	*0.51*/0.48	TL_1	TL_1	TL_1	TL_2
case179_goc_sad	1.12	0.95	**0.05**	0.93	*0.66*/0.42	TL_1	TL_1	4,414	TL_2
case200_activ_sad	0.03	**GOPT**	**GOPT**	**GOPT**	**GOPT**	2,187	805	53	0
case240_pserc_sad	4.93	3.42	4.34	3.16	*2.63*/**2.61**	TL_1	TL_1	27,875	TL_2
case300_ieee_sad	2.61	0.67	2.34	0.05	**GOPT**	TL_1	TL_1	5,654	0
case500_goc_sad	6.67	5.68	5.29	5.57	*5.21*/**5,18**	TL_1	TL_1	TL_1	TL_2

[1] IPOPT did not manage to solve the Determinant-SDP relaxation for the instance case240_pserc_api.

6 Conclusion and perspectives

We introduce a Conic Programming relaxation for the AC Optimal Power Flow problem. This relaxation is a tightening of the classical Semidefinite Programming relaxation with additional variables and valid inequalities. These inequalities dominate previously introduced nonlinear cuts, used to strengthen convex relaxations. Our numerical experiments on a reference benchmark illustrate that this Conic Programming relaxation is particularly suitable for a Bound Tightening procedure: it closes the gap in many cases where a Bound Tightening based on a Quadratic Convex relaxation does not. We also introduce an iterative scheme based on Mixed-Integer Linear Programming, that converges asymptotically towards global minimizers of the AC Optimal Power Flow problem. For the instances where the Bound Tightening procedure does not close the gap, this iterative scheme is able to reduce significantly the gap in most of the cases. A future line of research will consist in improving the scalability of the Optimization-Based Bound Tightening: parallelizing this procedure, or targeting the bounds to tighten, based on the graph structure. Another avenue to explore is the possibility of speeding-up the solution of the Mixed-Integer Linear Programming problem at a given iteration, by reusing the Branch-and-Bound trees of the problems solved during the previous iterations.

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A Strict dominance of Constraints (8)–(13) with respect to Constraints (†)–(‡)

In Section 3.2, Proposition 5 states that Constraints (8)–(13) dominate Constraints (†)–(‡). The advantage of Constraints (8)–(13) is to enforce a coupling between the convex envelopes of the quadruplets $(\mathsf{Re}(W_{ba}), \mathsf{Im}(W_{ba}), W_{bb}, W_{aa})$ involving the same index b. In this Appendix, we present an illustrative example of two quadruplets $(\mathsf{Re}(W_{ba}), \mathsf{Im}(W_{ba}), W_{bb}, W_{aa})$ and $(\mathsf{Re}(W_{bc}), \mathsf{Im}(W_{bc}), W_{bb}, W_{cc})$ satisfying Constraints (†)–(‡) introduced in [10], but for which there are no vectors L and R s.t. Constraints (8)–(13) are simultaneously satisfied for (b, a) and (b, c).

We consider any $(b, a, c) \in \mathcal{B}^3$ with the following realistic data:

- Voltage Magnitude Bounds: $\underline{v}_b = \underline{v}_a = \underline{v}_c = 0.9$ and $\overline{v}_b = \overline{v}_a = \overline{v}_c = 1.1$,
- Phase Angle Difference Bounds: $\overline{\theta}_{ba} = \overline{\theta}_{bc} = -\underline{\theta}_{ba} = -\underline{\theta}_{bc} = \arccos(0.99) \simeq 8.11°$.

As a consequence, we have $v^\sigma_b = v^\sigma_a = v^\sigma_c = 2$ and $\phi_{ba} = \phi_{bc} = 0$ and $\delta_{ba} = \delta_{bc} = \arccos(0.99)$. We consider the quadruplets

$$(\mathsf{Re}(W_{ba}), \mathsf{Im}(W_{ba}), W_{bb}, W_{aa}) = (1.1, 0, 1, 1.21) \tag{48}$$
$$(\mathsf{Re}(W_{bc}), \mathsf{Im}(W_{bc}), W_{bb}, W_{cc}) = (1.085, 0, 1, 1.21), \tag{49}$$

noticing that W_{bb} has indeed the same value in both quadruplets. These quadruplets both satisfy Equations (†)–(‡).

- First quadruplet: It satisfies (†), since $v^\sigma_b v^\sigma_a (\cos(\phi_{ba})\mathsf{Re}(W_{ba}) + \sin(\phi_{ba})\mathsf{Im}(W_{ba})) - \overline{v}_a \cos(\delta_{ba}) v^\sigma_a W_{bb} - \overline{v}_b \cos(\delta_{ba}) v^\sigma_b W_{aa} = 2 \times 2 \times 1 \times 1.1 + 0 - 1.1 \times 0.99 \times 2 \times 1 - 1.1 \times 0.99 \times 2 \times 1.21 = -0.41338$, which is greater than $\overline{v}_b\overline{v}_a \cos(\delta_{ba})(\underline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a) = 1.1 \times 1.1 \times 0.99 \times (0.9 \times 0.9 - 1.1 \times 1.1) = -0.47916$. It satisfies (‡), since $v^\sigma_b v^\sigma_a (\cos(\phi_{ba})\mathsf{Re}(W_{ba}) + \sin(\phi_{ba})\mathsf{Im}(W_{ba})) - \underline{v}_a \cos(\delta_{ba}) v^\sigma_a W_{bb} - \underline{v}_b \cos(\delta_{ba}) v^\sigma_b W_{aa} = 2 \times 2 \times 1 \times 1.1 + 0 - 0.9 \times 0.99 \times 2 \times 1 - 0.9 \times 0.99 \times 2 \times 1.21 = 0.46178$ is greater than $-\underline{v}_b\underline{v}_a \cos(\delta_{ba})(\underline{v}_b\underline{v}_a - \overline{v}_b\overline{v}_a) = -0.9 \times 0.9 \times 0.99 : \times (0.9 \times 0.9 - 1.1 \times 1.1) = 0.32076$.
- Second quadruplet: It satisfies (†), since $v^\sigma_b v^\sigma_c (\cos(\phi_{bc})\mathsf{Re}(W_{bc}) + \sin(\phi_{bc})\mathsf{Im}(W_{bc})) - \overline{v}_c \cos(\delta_{bc}) v^\sigma_c W_{bb} - \overline{v}_b \cos(\delta_{bc}) v^\sigma_b W_{cc} = 2 \times 2 \times 1 \times 1.085 + 0 - 1.1 \times 0.99 \times 2 \times 1 - 1.1 \times 0.99 \times 2 \times 1.21 = -0.47338$, which is greater than $\overline{v}_b\overline{v}_c \cos(\delta_{bc})(\underline{v}_b\underline{v}_c - \overline{v}_b\overline{v}_c) = 1.1 \times 1.1 \times 0.99 \times (0.9 \times 0.9 - 1.1 \times 1.1) = -0.47916$. It satisfies (‡), since $v^\sigma_b v^\sigma_c (\cos(\phi_{bc})\mathsf{Re}(W_{bc}) + \sin(\phi_{bc})\mathsf{Im}(W_{bc})) - \underline{v}_c \cos(\delta_{bc}) v^\sigma_c W_{bb} - \underline{v}_b \cos(\delta_{bc}) v^\sigma_b W_{cc} = 2 \times 2 \times 1 \times 1.085 + 0 - 0.9 \times 0.99 \times 2 \times 1 - 0.9 \times 0.99 \times 2 \times 1.21 = 0.40178$ is greater than $-\underline{v}_b\underline{v}_c \cos(\delta_{bc})(\underline{v}_b\underline{v}_c - \overline{v}_b\overline{v}_c) = -0.9 \times 0.9 \times 0.99 : \times (0.9 \times 0.9 - 1.1 \times 1.1) = 0.32076$.

We assume now that there exists $L_b \in [\underline{v}_b, \overline{v}_b], L_a \in [\underline{v}_a, \overline{v}_a], L_c \in [\underline{v}_c, \overline{v}_c], R_{ba} \in [\underline{v}_b\underline{v}_a, \overline{v}_b\overline{v}_a], R_{bc} \in [\underline{v}_b\underline{v}_c, \overline{v}_b\overline{v}_c]$ s.t. Constraints (8)–(13) are satisfied. Since $W_{aa} = \overline{v}_a^2$, we deduce from Constraint (11) that $\overline{v}_a^2 + \underline{v}_a\overline{v}_a =$

$R_{aa} + \underline{v}_a\bar{v}_a \leq (\underline{v}_a + \bar{v}_a)L_a$, i.e., that $(\underline{v}_a + \bar{v}_a)\bar{v}_a \leq (\underline{v}_a + \bar{v}_a)L_a$, and, thus, $\bar{v}_a \leq L_a$ since $(\underline{v}_a + \bar{v}_a) > 0$. As $\bar{v}_a \geq L_a$ by definition of L_a, we observe that $L_a = \bar{v}_a$. We use then $R_{ba} \leq \bar{v}_a L_b + \underline{v}_b L_a - \bar{v}_a \underline{v}_b$ from Constraint (9) to deduce that $R_{ba} \leq \bar{v}_a L_b$. Constraint (12) gives $|W_{ba}| \leq R_{ba}$, meaning that $L_b \geq \frac{|W_{ba}|}{\bar{v}_a} = 1$. As $L_b^2 \leq W_{bb} = 1$ according to Constraint (10), $L_b = 1$. As we did for a, we deduce from $W_{aa} = \bar{v}_a^2$ and Constraint (11) that $L_c = \bar{v}_c$. We use $R_{bc} \leq \bar{v}_c L_b + \underline{v}_b L_c - \bar{v}_c \underline{v}_b$ from Constraint (9) to state that $R_{bc} \leq \bar{v}_c L_b$, and we use $R_{bc} \geq \bar{v}_b L_c + \bar{v}_c L_b - \bar{v}_b \bar{v}_c$ to state that $R_{bc} \geq \bar{v}_c L_b$. Hence, $R_{bc} = \bar{v}_c L_b = 1.1$. As Constraint (13) imposes $\cos(\phi_{bc})\mathsf{Re}(W_{bc}) + \sin(\phi_{bc})\mathsf{Im}(W_{bc}) \geq R_{bc}\cos(\delta_{bc})$, we deduce that $(\phi_{bc})\mathsf{Re}(W_{bc}) + \sin(\phi_{bc})\mathsf{Im}(W_{bc}) \geq 1.089$. This is contradictory with the fact that $(\phi_{bc})\mathsf{Re}(W_{bc}) + \sin(\phi_{bc})\mathsf{Im}(W_{bc}) = \mathsf{Re}(W_{bc}) = 1.085$. As a conclusion, there does not exist $L_b \in [\underline{v}_b, \bar{v}_b], L_a \in [\underline{v}_a, \bar{v}_a], L_c \in [\underline{v}_c, \bar{v}_c], R_{ba} \in [\underline{v}_b \underline{v}_a, \bar{v}_b \bar{v}_a], R_{bc} \in [\underline{v}_b \underline{v}_c, \bar{v}_b \bar{v}_c]$ s.t. Constraints (8)–(13) are satisfied simultaneously for the pairs (b, a) and (b, c).

This illustrates the interest of setting the trigonometric cuts (13) with a variable radius R_{ba}, whereas previous works, to our knowledge, only use cuts associated to an extreme value of R_{ba}.

B Nonlinear but convex objective and constraints in relaxation $(\mathbf{R})$

We recall that the decision vector in relaxation $(\mathbf{R})$ is $x = (\mathsf{Re}(S), \mathsf{Im}(S), \mathsf{Re}(W), \mathsf{Im}(W), L, R)$. First, we denote by $\mathcal{P} \subset \mathbb{R}^N$ the polytope defined by the following box constraints:

- For all $g \in \mathcal{G}$, $\mathsf{Re}(S_g) \in [\mathsf{Re}(\underline{s}_g), \mathsf{Re}(\bar{s}_g)]$ and $\mathsf{Im}(S_g) \in [\mathsf{Im}(\underline{s}_g), \mathsf{Im}(\bar{s}_g)]$,
- For all $(b, a) \in \mathcal{E}$, $\mathsf{Re}(W_{ba}) \in [0, \bar{v}_b \bar{v}_a]$, $\mathsf{Im}(W_{ba}) \in [-\bar{v}_b \bar{v}_a, \bar{v}_b \bar{v}_a]$ and $R_{ba} \in [\underline{v}_b \underline{v}_a, \bar{v}_b \bar{v}_a]$,
- For all $b \in \mathcal{B}$, $L_b \in [\underline{v}_b, \bar{v}_b]$.

Now, we review the nonlinear terms in the objective and in the constraints of relaxation $(\mathbf{R})$, as functions of x. We show that all these functions have the form $f(x) = \max_{u \in \mathcal{U}} u^\top \pi(x)$ for all $x \in \mathcal{P}$, with a given affine application $\pi : \mathbb{R}^N \mapsto \mathbb{R}^p$ and a compact and convex set $\mathcal{U} \subset \mathbb{R}^p$.

- **The objective function** is $\sum_{g \in \mathcal{G}} \big(c_{1g}\,\mathsf{Re}(S_g) + \sum_{g \in \mathcal{G}_2} c_{2g}\,\mathsf{Re}(S_g)^2\big)$, where $\mathcal{G}_2$ is the set of generators $g \in \mathcal{G}$ s.t. $c_{2g} > 0$. This function reads $\max_{u \in \mathcal{U}} u^\top \pi(x)$ for all $x \in \mathcal{P}$ with
  - The compact and convex set $\mathcal{U} = \{1\} \times \prod_{g \in \mathcal{G}_2} \{(z_1, -z_2) \,:\, z_1^2 \leq z_2, z_1 \in [\mathsf{Re}(\underline{s}_g), \mathsf{Re}(\bar{s}_g)]\}$,
  - The affine application $\pi(x) = \big(\sum_{g \in \mathcal{G}} c_{1g}\mathsf{Re}(S_g), (2c_{2g}\mathsf{Re}(S_g), c_{2g})_{g \in \mathcal{G}_2}\big)$.
- **Thermal limits for lines** yield constraints with the form $|y_1^* W_{bb} + y_2^* W_{ba}| - \bar{S}_{ba} \leq 0$, with $(y_1, y_2) = (Y_{ba}^{\mathsf{ff}}, Y_{ba}^{\mathsf{ft}})$ if $(b, a) \in \mathcal{L}$ or $(y_1, y_2) = (Y_{ab}^{\mathsf{tt}}, Y_{ab}^{\mathsf{tf}})$ if $(b, a) \in \mathcal{L}^R$. Introducing $(r_1, h_1, r_2, h_2) = (\mathsf{Re}(y_1), \mathsf{Im}(y_1), \mathsf{Re}(y_2), \mathsf{Im}(y_2))$, this constraint is $\sqrt{(r_1\mathsf{Re}(W_{bb}) + r_2\mathsf{Re}(W_{ba}) + h_2\mathsf{Im}(W_{ba}))^2 + (-h_1\mathsf{Re}(W_{bb}) + r_2\mathsf{Im}(W_{ba}) - h_2\mathsf{Re}(W_{ba}))^2} - \bar{S}_{ba} \leq 0$. This is $\max_{u \in \mathcal{U}} u^\top \pi(x) \leq 0$ with
  - The compact and convex set $\mathcal{U} = \{(-1, z_1, z_2) \in \mathbb{R}^3 \,:\, z_1^2 + z_2^2 \leq 1\}$,
  - The affine application $\pi(x) = (\bar{S}_{ba}, r_1\mathsf{Re}(W_{bb}) + r_2\mathsf{Re}(W_{ba}) + h_2\mathsf{Im}(W_{ba}), -h_1\mathsf{Re}(W_{bb}) + r_2\mathsf{Im}(W_{ba}) - h_2\mathsf{Re}(W_{ba}))$.
- **Constraint** (10), i.e., $L_b^2 - R_{bb} \leq 0$ for any $b \in \mathcal{B}$, has the form $\max_{u \in \mathcal{U}} u^\top \pi(x) \leq 0$, with
  - The compact and convex set $\mathcal{U} = \{(-1, z_1, -z_2) \in \mathbb{R}^2 \,:\, z_1^2 \leq z_2 \in [\underline{v}_b, \bar{v}_b]\}$,
  - The affine application $\pi(x) = (R_{bb}, 2L_b, 1)$.
- **Constraint** (12), i.e., $|W_{ba}| - R_{ba} \leq 0$ for any $(b, a) \in \mathcal{E}$, has the form $\max_{u \in \mathcal{U}} u^\top \pi(x) \leq 0$ with
  - The compact and convex set $\mathcal{U} = \{(-1, z_1, z_2) \in \mathbb{R}^3 \,:\, z_1^2 + z_2^2 \leq 1\}$,
  - The affine application $\pi(x) = (R_{ba}, \mathsf{Re}(W_{ba}), \mathsf{Im}(W_{ba}))$.
- The relaxation $(\mathbf{R})$ includes several **SDP constraints** $\mathcal{A}(x) \succeq 0$, where $\mathcal{A}$ is a linear matrix operator. Such a constraint amounts to $\max_{u \in \mathcal{U}} u^\top \pi(x) \leq 0$ with
  - The compact and convex set $\mathcal{U} = \{M \in \mathbb{H}_p \,:\, (\mathsf{Tr}(M) = 1) \wedge (M \succeq 0)\}$,
  - The linear application $\pi(x) = -\mathcal{A}(x)$,

  and seeing $p \times p$ Hermitian matrices as real vectors of length $2p^2$.

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